The coefficients of exponential Fourier series of an *even* periodic waveform on *t* will be *real* and its trigonometric Fourier series will contain only *cosine* terms.

The coefficients of exponential Fourier series of an *odd* periodic waveform on *t* will be *imaginary* and its trigonometric Fourier series will contain only *sine* terms.

A periodic waveform with *half-wave symmetry* does not have any average value (or dc content) and does not contain any *even harmonics*.

If a periodic waveform is *even* on *t* and is *half-wave symmetric*, its Fourier series expansion will contain only *cosine functions* at *odd harmonic frequencies*.

If a periodic waveform is *odd* on *t* and is *half-wave symmetric*, its Fourier series expansion will contain only *sine functions* at *odd harmonic frequencies*.

$$\therefore \tilde{v}_n = \begin{cases} 0 \text{ for even } n \\ \dfrac{2}{T}\displaystyle\int_0^{\frac{1}{2}T} v(t)e^{-jn\omega_o t}\,dt \text{ for odd } n \end{cases}$$

$$a_o = 0$$

$$a_n = \begin{cases} 0 \text{ for even } n \\ \dfrac{4}{T}\displaystyle\int_0^{\frac{1}{2}T} v(t)\cos n\omega_o t\,dt \text{ for odd } n \end{cases} \text{ and } b_n = \begin{cases} 0 \text{ for even } n \\ \dfrac{4}{T}\displaystyle\int_0^{\frac{1}{2}T} v(t)\sin n\omega_o t\,dt \text{ for odd } n \end{cases}$$

13.6 Properties of Fourier Series and Some Examples

We take up some examples of Fourier series and develop the important properties of Fourier series through them. The waveforms used in the examples that follow are very important signal waveforms that appear frequently in Electrical and Electronic Engineering applications and are not just some functions used to illustrate Fourier series. The waveforms appearing in the examples are important in their own right.

The first property of Fourier series is almost self-evident. It is the property of *linearity*. It states that if $v_1(t)$ and $v_2(t)$ are two periodic waveforms with same period T and $v_3(t) = a_1 v_1(t) + a_2 v_2(t)$, then,

$\tilde{v}_{3n} = a_1\tilde{v}_{1n} + a_2\tilde{v}_{2n}$ for all n where $\tilde{v}_{1n}, \tilde{v}_{2n}$ and $\tilde{v}_{3n}$ are the coefficients of exponential Fourier series of $v_1(t)$, $v_2(t)$ and $v_3(t)$ respectively. This may be proved easily.

Example : 13.6-1

Find the exponential Fourier series of $v(t) = \sum_{k=-\infty}^{\infty}\delta(t-k)$ and derive the trigonometric Fourier series from it.

Solution

This waveform is a periodic sequence of unit impulses with a period of 1 sec. It is shown in Fig. 13.6-1.

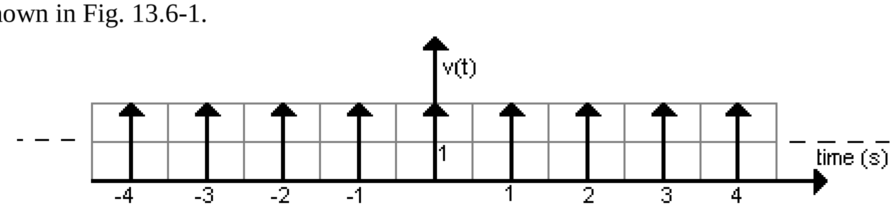

Fig. 13.6-1 Waveform $v(t)$ in Example : 13.6-1

$$\tilde{v}_n = \frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} v(t)e^{-jn\omega_o t}\,dt = \frac{1}{1}\int_{-\frac{1}{2}}^{\frac{1}{2}} v(t)e^{-jn2\pi t}\,dt = \int_{0^-}^{0^+}\delta(t)e^{-jn2\pi t}\,dt = \int_{0^-}^{0^+}\delta(t)\,dt = 1$$

The waveform has zero value at all points in the first period [-0.5,0.5] except between 0^- and 0^+. In that interval it is an impulse of unit magnitude. And the value of exponential in that interval is 1. Therefore, all coefficients in the exponential Fourier series are equal to 1.

$$\therefore \sum_{k=-\infty}^{\infty}\delta(t-k) = \sum_{n=-\infty}^{\infty} e^{j2\pi nt} \text{ is the exponential Fourier series of } v(t).$$

We obtain the trigonometric Fourier series from exponential Fourier series by taking two components with harmonic order $-n$ and $+n$ together.

$$\therefore \sum_{k=-\infty}^{\infty} \delta(t-k) = \sum_{n=-\infty}^{\infty} e^{j2\pi nt} = e^{j2\pi .0.t} + \sum_{n=1}^{\infty}(e^{j2\pi nt} + e^{-j2\pi nt}) = 1 + \sum_{n=1}^{\infty} 2\cos 2\pi nt \quad (13.6\text{-}1)$$

The waveform contains a dc component of 1 unit. This should be so since the total area content of the waveform in one period is the area content of unit impulse, *i.e.*, unity. This area divided by *T* must be the dc content in the waveform. *T* in the example is 1 sec.

The waveform contains only cosine terms. This too is expected since $v(t)$ is an *even* function of *t*.

But most important aspect to be noted is that periodic impulse train contains *all harmonics with equal strength*. That is, the amplitude of harmonic components does not show any let up as the frequency of harmonic component goes up. Sinusoids of all frequencies with uniform amplitude are required to synthesize the periodic impulse train.

A periodic impulse train contains *all harmonics with equal strength*.

But didn't we stretch the concept of Fourier series a bit too far? Impulse is a highly discontinuous waveform. In fact, the $v(t)$ in this example violates the Dirichlet's condition that, discontinuity , if present, must be of finite value. Hence, though we have a found a Fourier series for $v(t)$, will it really converge to $v(t)$ at all *t* ?

The answer, strictly speaking, is that, it will not. As usual, faced with such mathematical difficulties, we just change our viewpoint and make the Fourier series we derived in this example a useful one! We simply view the impulse as rectangular pulse of large height and small width and unit area. Then we argue that over the width of rectangular pulse in the first period (which is centered around t=0) the exponential factor $e^{-jk\omega_o t}$ is close to 1 and may be approximated as such. But won't the approximation fail if *k* becomes very large though *t* is small? It may, but we will not let it fail; we will state that we will compress the pulse a little more while keeping its area at unity! Hence the Fourier series in Eqn. 13.6-1 represents the Fourier series of a periodic rectangular pulse train with each pulse containing unit area as the width of the pulse is made infinitesimal and height of the pulse is made infinitely large. We keep Dirichlet happy that way!

Example : 13.6-2

Obtain the Fourier series of $v(t) = \sum_{n=-\infty}^{\infty} \delta(t-n-0.5)$ shown in Fig. 13.6-2.

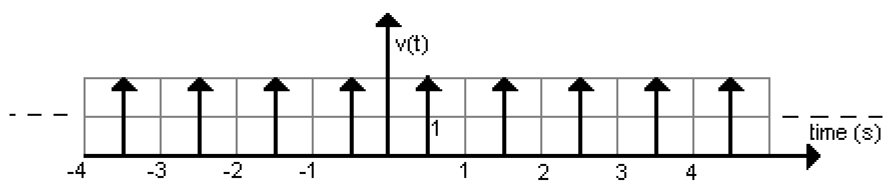

Fig. 13.6-2 Waveform *v*(*t*) for Example : 13.6-2

Solution

We observe by comparing Fig. 13.6-1 and Fig. 13.6-2 that this waveform is only a *delayed* version of waveform in Fig. 13.6-1. The delay involved is 0.5 sec. Therefore if we delay all the sinusoidal components in the Fourier series of waveform in Fig. 13.6-1 we should get the Fourier series of waveform in Fig. 13.6-2. Delaying a sinusoid by t_d seconds amounts to adding a phase delay of $\omega\, t_d$ radians to its argument where ω is its radian frequency. That is, ωt has to be replaced by $(\omega t - \phi)$ where $\phi = \omega\, t_d$.

k^{th} harmonic component in the exponential Fourier series of any waveform is denoted by $\tilde{v}_k e^{jk\omega_o t}$ where ω_o is the fundamental radian frequency. Delaying this component by t_d sec results in $\tilde{v}_n e^{j(n\omega_o t - n\omega_o t_d)} = [\tilde{v}_n e^{-jn\omega_o t_d}]e^{jn\omega_o t}$. Therefore, the

Fourier series of a time-shifted waveform can be expressed in terms of Fourier series of the unshifted version as follows.

Time-shifting property of Fourier series

If $\tilde{v}_n$ are the coefficients of exponential Fourier series of a periodic waveform v(t), then, the coefficients of exponential Fourier series of the time-shifted periodic waveform $v(t-t_d)$ are given by $\tilde{v}_n e^{-jn\omega_o t_d}$. This is called the 'Time-shifting property of Fourier series expansion'.

In this example, the time delay is 0.5 sec and that value is half the period. Therefore, $e^{-jn\omega_o t_d} = e^{-j\pi n} = 1$ for even n and -1 for odd n.

Therefore Fourier series of $v(t)$ is as shown below.

$$\therefore \sum_{k=-\infty}^{\infty} \delta(t-k-0.5) = \sum_{n=-\infty}^{\infty} e^{-j\pi n} e^{j2\pi nt} = 1 + \sum_{n=1}^{\infty} (-1)^n 2\cos 2\pi nt \qquad (13.6\text{-}2)$$

There is no change in the amplitude of cosine waves, but the phase of waves alternate between 0 and 180^0 with harmonic order n.

Example : 13.6-3

Find the exponential Fourier series and trigonometric Fourier series of waveforms in Example : 13.6-1 and Example : 13.6-2 if the magnitude of each impulse in both cases is V units and the period is T sec.

Solution

We remember that there is a $1/T$ factor in the Fourier series analysis equation. It happened to be 1 when T was set to 1 sec. We have to bring that factor back. Similarly, since $\omega_o = \frac{2\pi}{T}$ we have to bring in $1/T$ in the index of exponential in exponential Fourier series and in the argument of trigonometric functions in trigonometric Fourier series.

Hence the required Fourier series for waveform in Example : 13.6-1 is

$$\therefore \sum_{k=-\infty}^{\infty} V\delta(t-kT) = \sum_{n=-\infty}^{\infty} \frac{V}{T} e^{j\frac{2\pi}{T}nt} = \frac{V}{T} + \sum_{n=1}^{\infty} \frac{2V}{T}\cos\frac{2\pi}{T}nt$$

and the required Fourier series for waveform in Example : 13.6-2 is

$$\therefore \sum_{k=-\infty}^{\infty} V\delta(t-kT-\tfrac{T}{2}) = \sum_{n=-\infty}^{\infty} \left(\frac{V}{T}e^{-j\pi n}\right) e^{j\frac{2\pi}{T}nt} = \frac{V}{T} + \sum_{n=1}^{\infty} \frac{2V}{T}(-1)^n \cos\frac{2\pi}{T}nt$$

Example : 13.6-4

Let $v_1(t)$ be a periodic waveform same as the one used in Example : 13.6-1 and $v_2(t)$ be a periodic waveform same as the waveform used in Example : 13.6-2. Then, find $v(t) = v_1(t) - v_2(t)$ and its Fourier series.

Solution

The waveform $v(t)$ is constructed and shown in Fig. 13.6-3

We construct the Fourier series of this waveform by using Eqn. 13.6-1 and Eqn. 13.6-2 and property of linearity of Fourier series.

$$\therefore v(t) = \sum_{n=-\infty}^{\infty} e^{j2\pi nt} - \sum_{n=-\infty}^{\infty} e^{-jn\pi} e^{j2\pi nt} = \sum_{\substack{n=-\infty \\ odd\ n}}^{\infty} 2e^{j2\pi nt} = \sum_{\substack{n=1 \\ odd\ n}}^{\infty} 4\cos 2\pi nt$$

It contains only odd harmonics of cosine format. Its average value is zero. It should be so since this waveform has *even* symmetry and *half-wave* symmetry.

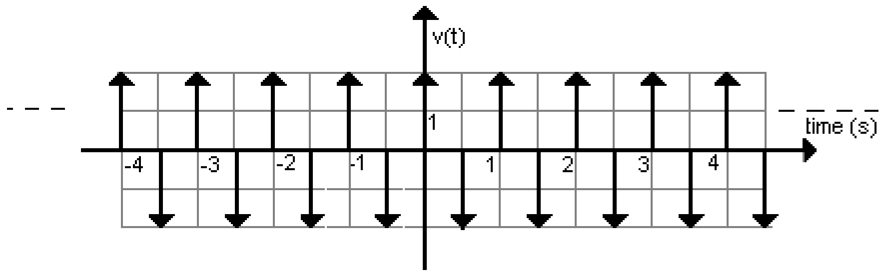

Fig. 13.6-3 Waveform of $v(t)$ in Example : 13.6-4

Example : 13.6-5

$v(t)$ is a periodic waveform with same waveform as that of Fig. 13.6-3 , but with 2 units of magnitude in each impulse. Find and plot $v_i(t) = \int_{-\infty}^{t} v(t)\,dt$ and obtain Fourier series of $v_i(t)$.

Solution

Integration usually results in an arbitrary constant. This constant will become a dc component in $v_i(t)$. Since there is no way to find out this arbitrary constant we choose to ignore it and we qualify the Fourier series of $v_i(t)$ by adding a clause that a dc term can always come in the series without upsetting other coefficients. The waveform of $v_i(t)$, ignoring a possible dc term, is shown in Fig. 13.6-4 .

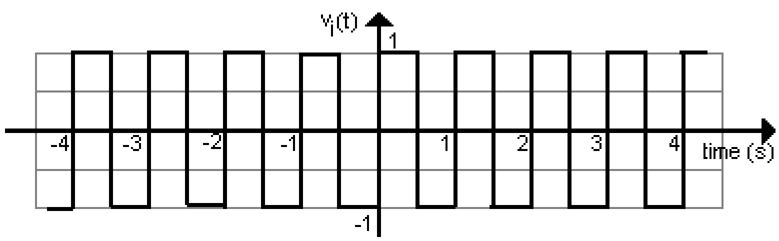

Fig. 13.6-4 Waveform of $v_i(t)$ in Example : 13.6-5

The dc content was ignored. Therefore $v_i(t)$ must be pure ac. At t = 0 its value will change by the area content of impulse at that point. This area content is 2 units. Hence the value of $v_i(t)$ must change *by* 2 units at t = 0. After that it must remain constant. At t = 0.5 s the negative impulse of 2 unit magnitude will again change $v_i(t)$ by –2 units. With the assumption of ac output, $v_i(t)$ must then be alternating between +1 and –1 with a period of 1 sec. Hence $v_i(t)$ is a symmetric square wave of amplitude 1 unit and period 1 sec.

We wish to find the Fourier series of $v_i(t)$. There are two ways to do it. The first method is to employ the analysis equation of Fourier series- Eqn. 13.2-2 –and carry out the required integration to determine the Fourier series coefficients. The second method relies upon the fact the waveform $v_i(t)$ is the integral of $v(t)$ and that we already know the Fourier series of $v(t)$. That leads us to the question – what is the relationship between Fourier series coefficients for a periodic waveform and Fourier series coefficients for its integral?

$$v_i(t) = \int_{-\infty}^{t} v_i(t)\,dt = \int_{-\infty}^{t} \sum_{n=-\infty}^{\infty} \tilde{v}_n e^{jn\omega_o t} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{t} \tilde{v}_n e^{jn\omega_o t} dt = \sum_{n=-\infty}^{\infty} \frac{\tilde{v}_n}{jn\omega_o} e^{jn\omega_o t} \qquad (13.6\text{-}3)$$

$$\therefore \tilde{v}_{in} = \frac{\tilde{v}_n}{jn\omega_o}$$

Note that $v(t)$ should not have an average value, *i.e.*, $\tilde{v}_0 = a_0 = 0$. If it has a non-zero average value, then its integral will have a linearly increasing term which makes it aperiodic and unbounded. Fourier series for such a waveform does not exist.

Integration in Time property of Fourier series

We have arrived at the *Integration in Time* property of Fourier series.

If v(t) is a zero-average periodic waveform with $\tilde{v}_n$ *as its exponential Fourier series coefficients, then the Fourier series coefficients of* $\int_{-\infty}^{t} v(t)dt$ *is given by* $\frac{\tilde{v}_n}{jn\omega_o}$.

The Fourier series coefficients of half of $v(t)$ in this example was derived in Example : 13.6-4.

$$v(t) = \sum_{\substack{n=-\infty \\ odd\ n}}^{\infty} 4e^{j2\pi nt} = \sum_{\substack{n=1 \\ odd\ n}}^{\infty} 8\cos 2\pi nt$$

$$\therefore v_i(t) = \sum_{\substack{n=-\infty \\ odd\ n}}^{\infty} \frac{4}{j2\pi n} e^{j2\pi nt} = \sum_{\substack{n=1 \\ odd\ n}}^{\infty} \frac{2}{j\pi n} e^{j2\pi nt} + \frac{2}{j\pi(-n)} e^{j2\pi(-n)t} = \sum_{\substack{n=1 \\ odd\ n}}^{\infty} \frac{4}{\pi}\frac{1}{n}\sin 2\pi nt$$

A symmetric square wave contains only odd harmonics and the harmonic amplitude (= $\frac{4}{\pi n}$ **in trigonometric Fourier series) decreases in inverse proportion to harmonic order *n*.**

A unit amplitude symmetric square wave of period T will also have same coefficients, *i.e.,* $\frac{4}{\pi n}$. The factor of $\frac{1}{T}$ gets cancelled by a similar factor that is included in ω_o in Eqn. 13.6-3.

Example : 13.6-6

$v(t)$ is a ± 4 unit symmetric square wave with T = 1 sec. Find and plot $v_i(t) = \int_{-\infty}^{t} v(t)\,dt$ and find its Fourier series.

Solution

$v(t)$ and $v_i(t)$ are shown in Fig. 13.6-5.

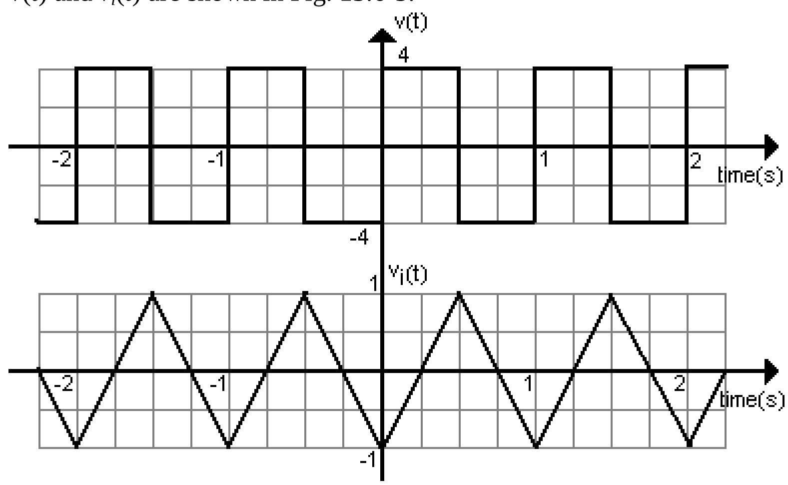

Fig. 13.6-5 Waveforms of $v(t)$ and its Integral, $v_i(t)$, in Example : 13.6-6

Using the previous result from the example, we get,

$$v(t) = \sum_{\substack{n=-\infty \\ odd\ n}}^{\infty} \frac{8}{j\pi n} e^{j2\pi nt} = \sum_{\substack{n=1 \\ odd\ n}}^{\infty} \frac{16}{\pi n}\sin 2\pi nt\ .$$

The exponential Fourier series coefficients get divided by $jn\omega_o$ when the waveform gets integrated in time.

$$\therefore v_i(t) = \sum_{\substack{n=-\infty \\ odd\ n}}^{\infty} \frac{16}{(j2\pi n)^2} e^{j2\pi nt} = \sum_{\substack{n=1 \\ odd\ n}}^{\infty} -\frac{4}{\pi^2 n^2}\left(e^{j2\pi nt} + e^{-j2\pi nt}\right) = \sum_{\substack{n=1 \\ odd\ n}}^{\infty} -\frac{8}{\pi^2 n^2}\cos 2\pi nt$$

A symmetric triangle wave contains only odd harmonics and the harmonic amplitude (= $\frac{8}{\pi^2 n^2}$ **in trigonometric Fourier series) decreases in inverse proportion to square of harmonic order *n*.**

The triangle wave in Fig. 13.6-5 has ±1 unit amplitude and has *even half-wave* symmetry. Hence it contains odd cosine harmonics.

Example : 13.6-7

Let $v(t)$ be a periodic waveform with period T and let $v_C(t)$ be defined as $v_c(t) = v(\alpha t)$ where $\alpha > 0$. Show that $v(t)$ and $v_C(t)$ have same coefficients in their Fourier series.

Solution

$v_C(t)$ will be a time-compressed version of $v(t)$ for α >1and time-expanded version of $v(t)$ for $\alpha < 1$. Therefore period of $v_C(t)$ will be $\frac{T}{\alpha}$. Let $\tilde{v}_{cn}$ and $\tilde{v}_n$ be the exponential Fourier series coefficients of $v_C(t)$ and $v(t)$ respectively.

$$
\begin{aligned}
\tilde{v}_{cn} &= \frac{\alpha}{T}\int_{-\frac{1}{2}\frac{\alpha}{T}}^{\frac{1}{2}\frac{\alpha}{T}} v_c(t) e^{j\frac{2\alpha\pi}{T}nt} dt \\
&= \frac{\alpha}{T}\int_{-\frac{1}{2}\frac{\alpha}{T}}^{\frac{1}{2}\frac{\alpha}{T}} v(\alpha t) e^{j\frac{2\pi}{T}n(\alpha t)} dt \\
&= \frac{\alpha}{T}\int_{-\frac{1}{2}\frac{\alpha}{T}}^{\frac{1}{2}\frac{\alpha}{T}} v(\alpha t) e^{j\frac{2\pi}{T}n(\alpha t)} \frac{1}{\alpha} d(\alpha t) \\
&= \frac{1}{T}\int_{-\frac{1}{2}\frac{\alpha}{T}}^{\frac{1}{2}\frac{\alpha}{T}} v(\alpha t) e^{j\frac{2\pi}{T}n(\alpha t)} d(\alpha t) \\
&= \tilde{v}_n
\end{aligned}
$$

$\therefore$ *Compression or expansion of a periodic waveform in time does not change its Fourier series coefficients. But fundamental frequency and harmonic frequency values will change. This property is called Time-scaling property of Fourier series.*

Time-scaling property of Fourier series

Note that *time-scaling* is not the same as a simple change in T alone. Let $v(t)$ a periodic train of rectangular pulses of unit amplitude and 0.1 sec width repeating with a period of 1 sec. Then $v(0.5t)$ is a periodic train of rectangular pulses of unit amplitude and 0.2 sec repeating with a period of 2 sec. Its Fourier series coefficients will be same as that of $v(t)$; but its fundamental frequency will be π rad/sec whereas that of $v(t)$ will be 2π rad/sec.

However, consider another periodic train of rectangular pulses of unit height and 0.1 sec width repeating with a period of 2sec. This is not the same as $v(0.5t)$. Its Fourier series coefficients will be different from that of $v(t)$. In fact, all Fourier series coefficients will get multiplied by $\frac{1}{2}$.

Example : 13.6-8

Find the Fourier series of the periodic rectangular pulse train shown in Fig. 13.6-6.

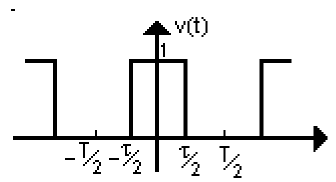

Fig. 13.6-6 Waveform for Example : 13.6-8

Solution

$$\tilde{v}_n = \frac{1}{T}\int_{-T/2}^{T/2} v(t) e^{-jn\omega_o t} dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} e^{-jn\omega_o t} dt = \frac{-1}{jn\omega_o T}\left[e^{-j n\omega_o\tau/2} - e^{j n\omega_o\tau/2}\right]$$

$$e^{-jn\omega_o\tau/2} - e^{jn\omega_o\tau/2} = -2j\sin \frac{n\omega_o\tau}{2} \text{ by Euler's Formula}$$

$$\therefore \tilde{v}_n = \frac{\tau}{T}\frac{2\sin \frac{n\omega_o\tau}{2}}{n\omega_o T\tau} = \frac{\tau}{T}\frac{\sin \frac{n\omega_o\tau}{2}}{\frac{n\omega_o\tau}{2}} = \left(\frac{\tau}{T}\right)\left(\frac{\sin x}{x}\right) \text{ where } x = \frac{n\omega_o\tau}{2}$$

The dc content of the waveform is $\frac{\tau}{T}$. The Fourier series contains only cosine terms since exponential Fourier series coefficients are real.

Example : 13.6-9

Fig. 13.6-7 shows the output of an absolute value circuit when input is a sinusoidal wave. This is the shape of output voltage of a full-bridge diode rectifier routinely used in ac-dc conversion applications. We wish to obtain the Fourier series for this waveshape. We bring out an important property of Fourier series first and use that property to obtain the required Fourier series in this example.

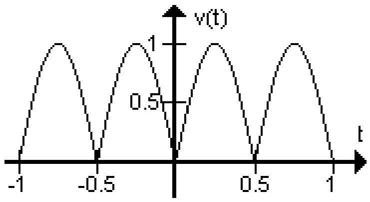

Fig. 13.6-7 Full-Wave Rectified Waveform in Example : 13.6-9

Solution

The waveform $v(t)$ is visualized first as the product of two waveforms as in Fig. 13.6-8.

The product of $v_2(t)$ which is a unit amplitude square wave with $v_1(t)$ which is a pure sine wave results in $v(t)$, the full-wave rectified waveform.

That raises the question – what are the exponential Fourier series coefficients of a product of two waveforms with same period in terms of exponential Fourier series coefficients of the constituent waveforms?

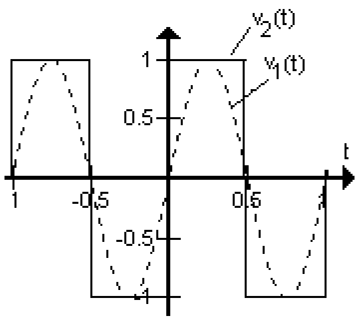

Fig. 13.6-8 Two Waveforms in a Product Results in *v*(*t*) of Example : 13.6-9

$$v_1(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} e^{jn\omega_o t} \quad \text{and} \quad v_2(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_{2n} e^{jn\omega_o t}$$

$$v(t) = v_1(t) v_2(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_k e^{jk\omega_o t}$$

Consider k^{th} component in the exponential Fourier series of $v(t)$. The waveform contributed to $v(t)$ by this component is $\tilde{v}_k e^{jk\omega_o t}$. Since $v(t)$ is the product of $v_1(t)$ and $v_2(t)$, this contribution can come up in $v(t)$ due to products of n^{th} contribution $\tilde{v}_{1n} e^{jn\omega_o t}$ in $v_1(t)$ and $(k\text{-}n)^{th}$ contribution $\tilde{v}_{2(k-n)} e^{j(k-n)\omega_o t}$ with n varying from $-\infty$ to $+\infty$.

$$\therefore \tilde{v}_k = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2(k-n)}$$

This is the so-called *Multiplication in Time* property of Fourier series.

Multiplication in Time property of Fourier series

If $v_1(t)$ and $v_2(t)$ are two periodic waveforms with same period and $v(t) = v_1(t) \times v_2(t)$, then, the exponential Fourier series coefficients of $v(t)$ is given by $\tilde{v}_k = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2(k-n)}$ for $-\infty < k < \infty$ *where* $\tilde{v}_{1n}$ and $\tilde{v}_{2n}$ *are the exponential Fourier series coefficients of $v_1(t)$ and $v_2(t)$ respectively.*

$v_1(t)$ is a sine wave in this example.

$$v_1(t) = \sin 2\pi t$$
$$= \frac{e^{j2\pi t} - e^{-j2\pi t}}{2j} \text{ (By Euler's Formula)}$$

$$\therefore \tilde{v}_1 = \frac{1}{2j}, \tilde{v}_{-1} = \frac{-1}{2j} \text{ and } \tilde{v}_n = 0 \text{ for all other values of } n$$

$v_2(t)$ is a unit amplitude square wave. Its Fourier series was obtained in Example : 13.6-5 as

$$\tilde{v}_{2n} = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{2}{j\pi n} e^{j2\pi nt}$$

$$\therefore \tilde{v}_k = \sum_{n=-\infty}^{\infty} \tilde{v}_{1n} \tilde{v}_{2(k-n)} = -\frac{1}{2j} \frac{2}{j\pi(k+1)} + \frac{1}{2j} \frac{2}{j\pi(k-1)} \text{ for even } k$$
$$= \frac{-2}{\pi(k^2-1)} \text{ for even } k$$

Since n takes only two values –1 and 1, and square wave has only odd harmonics, (k+1) and (k-1) have to be odd for $\tilde{v}_k$ to be non-zero.

$\therefore v(t)=\sum_{\substack{k=-\infty \\ \text{even } k}}^{\infty} \frac{-2}{\pi(k^2-1)} e^{j2\pi kt} = \frac{2}{\pi} - \frac{4}{\pi} \sum_{\substack{k=1 \\ \text{even } k}}^{\infty} \frac{1}{(k^2-1)} \cos 2\pi kt$ is the Fourier series of a full-wave rectified sinusoid of unit amplitude.

The average value of rectified waveform is $\frac{2}{\pi}$ or $\frac{2V_m}{\pi}$ in general where V_m is the peak value of sine wave undergoing rectification. It has only even cosine harmonics in it. What kind of symmetry is responsible for this? It is *even* function and hence only cosines are expected. Notice that the fundamental period of the waveform is 0.5 sec and *not* 1 sec. This 1 sec period comes up only in the square waveform and sine waveform we used to form the product. This is why there is no component at odd harmonic order in the rectified output. If we treat it as a waveform with 0.5-sec period, all odd and even harmonics of its fundamental frequency are present in it. Its fundamental frequency will be double the fundamental frequency of the sinusoid that underwent rectification.

13.7 Discrete Magnitude and Phase Spectrum

Spectral plot for a time-domain waveform displays the Fourier series coefficients graphically against frequency. Since the frequencies involved in a Fourier series are discrete values (fundamental frequency and its multiples), a plot of Fourier series coefficients can not be a continuous curve. Therefore, the spectral plot is called a *discrete spectrum*. The exponential Fourier series coefficients are complex in general and two plots – one for magnitude of coefficients and the other for phase angle of the coefficients – will be needed.

Discrete spectrum defined

The information on coefficients is portrayed as a series of vertical lines located at harmonic frequencies. These lines will be equidistant and the length of the lines will be proportional to magnitude of the coefficient in the case of magnitude spectrum and to phase in the phase spectrum. The harmonic order n is also used in the abscissa instead of ω or f. The discrete spectral plots of the unit amplitude square wave we covered in Example : 13.6-5 is shown in Fig. 13.7-1 for illustration. Its exponential Fourier series is

$$\tilde{v}_n = \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{2}{j\pi n} e^{j2\pi nt} .$$

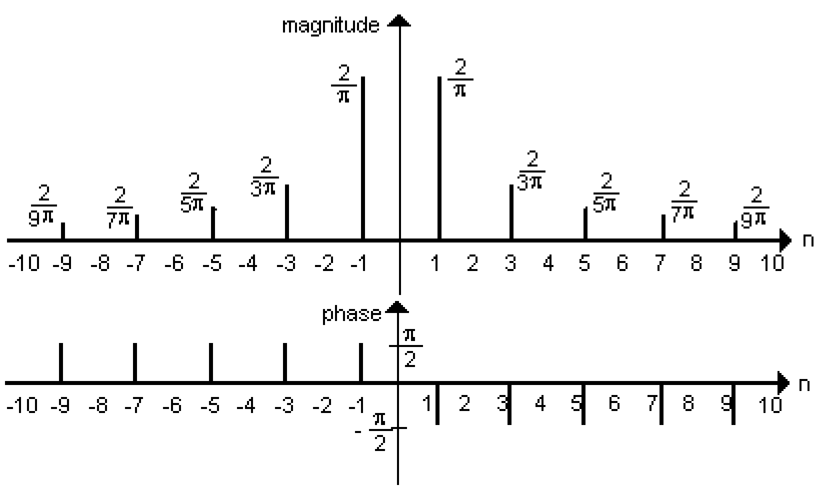

Fig. 13.7-1 Discrete Magnitude Spectrum and Phase Spectrum for a ±1 Square Wave Against Harmonic Order

The Fourier series coefficients of exponential Fourier series were plotted in the spectrum and that results in the so-called *two-sided spectrum.* It has been pointed out in earlier discussion that the two companion components at n and $-n$ always go together in exponential Fourier series. Two such components will add up to yield a *real* sinusoid always. They can not be split.

They always stay together.....

That the two components similarly placed on the left and right of origin in a two-sided spectrum should be viewed as an integral unit rather than as two separate components is to be kept in mind especially when interpreting two-sided spectral plots drawn against ω. If we forget that, we will be tempted to ask that often repeated question – *what is the meaning of negative frequency*?

Negative frequency ?

There is no negative *cyclic frequency*. There is no negative *angular frequency*. There are only two complex exponential functions - $e^{j\omega t}$ *and* $e^{-j\omega t}$. These two always get scaled by complex conjugate numbers and enter into a sum. They never appear individually once the circuit problem has been solved. They always go together and produce either a $\sin\omega t$ or a $\cos\omega t$ or a mixture of the two. Whatever they produce at the end will have an angular frequency of ω rad/sec and a cyclic frequency of $\omega/2\pi$ Hz.

No linear physical electrical circuit can ever do any processing on $e^{j\omega t}$ without carrying out the same processing on $e^{-j\omega t}$.

Both of them are complex exponential functions of time. Hence they have real and imaginary parts. Both, real and imaginary parts, are sinusoids. Those sinusoids have angular frequency of ω rad/sec and cyclic frequency of $\omega/2\pi$ Hz whether they come from $e^{j\omega t}$ or from $e^{-j\omega t}$. Therefore, *there is no negative radian frequency or cyclic frequency*.

However, we want to represent the magnitudes of scaling factors of $e^{j\omega t}$ *and* $e^{-j\omega t}$ and phases of scaling factors separately in a spectral plot. Therefore, as a part of notation for presenting information efficiently, we decide to extend the ω-axis to the left and put the data on scaling factor of $e^{-j\omega t}$ there. That does not make a value on the left side of ω-axis a *negative frequency*.

Note that the magnitude spectrum of a real $v(t)$ has to be necessarily *even* on ω and its phase spectrum has to be necessarily *odd* on ω. (Why?)

Example : 13.7-1

Some desktop off-line UPS (Uninterruptible Power Supply) units used for supplying single PC units deliver the waveform shown in Fig. 13.7-2 instead of a sine wave. (a) Find α if the third harmonic content is to become zero. (b) With this value of α, find V such that the r.m.s voltage is 220V. (c) Plot the magnitude and phase spectra with this value of α and V. (d) The purity of a sine wave is measured in terms of a quantity called 'Total Harmonic Distortion (THD)'.It is usually quoted in percentage and is defined as below.

$$THD=\frac{\sqrt{\sum_{n=2}^{\infty}\left|\tilde{v}_n\right|^2}}{\left|\tilde{v}_1\right|}\times 100\%$$

where $\tilde{v}_n$ is the exponential Fourier series coefficient. The r.m.s value of all harmonic components together is expressed as a percentage of r.m.s value of fundamental component in the THD measure. Amplitudes may be used instead of r.m.s values since it is a ratio. Calculate the THD of the waveform in this example.

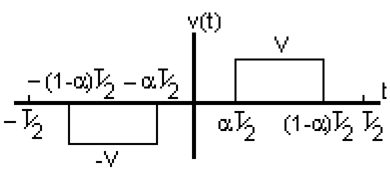

Fig. 13.7-2 Waveform for Example : 13.7-1

Solution

The trigonometric Fourier series of the waveform with V =1 and T = 1 is determined first. It is an *odd half-wave symmetric* waveform. Its trigonometric Fourier series will contain only odd sine harmonics. With V = 1 and T = 1,

$$\therefore v(t)=\sum_{n=1}^{\infty} b_n \sin n\omega_o t \text{ and } b_n=\frac{4}{T}\int_0^{T/2} v(t)\sin n\omega_o t\, dt$$

$$b_n=4\int_{\alpha/2}^{(1-\alpha)/2}\sin 2\pi nt\, dt$$

$$=\frac{4}{2\pi n}\left[\cos n\pi\alpha-\cos n\pi(1-\alpha)\right]$$

$$=\frac{4}{\pi n}\sin\frac{n\pi}{2}\sin\frac{n\pi(1-2\alpha)}{2}$$

The trigonometric Fourier series coefficients go to zero for even n as expected. Further, the Fourier series of this waveform approach that of a unit amplitude square wave as $\alpha \to 0$ as expected.

(a) If the third harmonic content is to be zero $\sin\frac{3\pi(1-2\alpha)}{2}$ must become zero. Therefore $\alpha = \frac{1}{6}$. With this value of α, the waveform will be zero for 1/3rd of a half cycle.

(b) Therefore r.m.s value = $\sqrt{\frac{2}{3}V^2} = 0.8165V$. If this is to be 220 volts, V must be ≈270 volts.

(c) The exponential Fourier series of $v(t)$ can be constructed from trigonometric Fourier series by noting that coefficients of sine terms will be twice the negative of imaginary part of exponential Fourier series coefficients. Therefore, with V=1 , T = 1 and α = 1/6,

$$\therefore v(t) = \sum_{\substack{n=-\infty \\ odd\ n}}^{\infty} -\frac{j2}{\pi n}\sin\frac{n\pi}{2}\sin\frac{n\pi}{3}e^{j2\pi nt} \qquad (13.7\text{-}1)$$

The two-sided spectrum is plotted in Fig. 13.7-3 with the scaling factor of 270 volts incorporated.

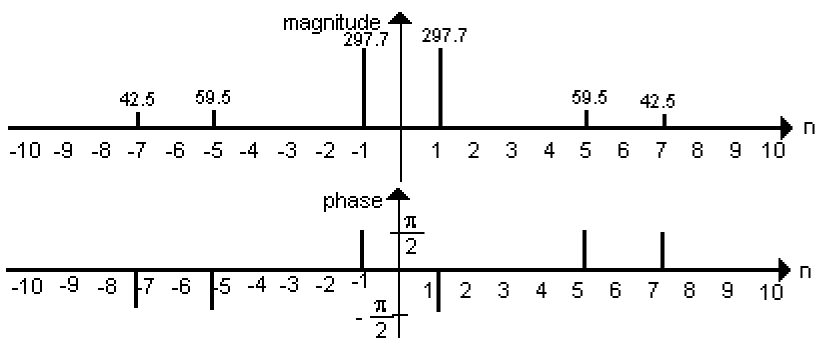

Fig. 13.7-3 Spectral Plots for $v(t)$ in Example : 13.7-1

(d) Refer to Eqn. 13.7-1. $\sin\frac{n\pi}{2}$ has a magnitude of 1 for all odd n. $\sin\frac{n\pi}{3}$ has a magnitude of 0 for all odd multiples of 3 and 0.866 for all other odd n including n = 1. $\frac{8}{\pi}$ is a common factor.

Therefore THD = $\sqrt{\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{17^2}+\frac{1}{19^2}+\ldots} \times 100 \approx 28.5\%$

13.8 Rate of Decay of Harmonic Amplitude

Fourier series is an infinite series. It requires infinite number of sinusoids with frequencies ranging from fundamental frequency to infinitely large frequency to synthesize a non-sinusoidal periodic waveform in general. There may be special cases where the Fourier series terminates at some finite harmonic order. But they are only special cases.

This indicates that we have to find out the ac steady-state response of the circuit to each and every component in Fourier series of input and sum up them to get the periodic steady-state response of the circuit. That calls for infinite computation – we will not get done with it. Hence the issue of rate of decay of harmonic amplitudes is of practical significance in deciding how many terms from the Fourier series should we carry in any analysis problem.

Harmonic amplitude varies with harmonic order in general. Information on rate of variation is needed in analysis and design of circuits dealing with switched waveforms.

Circuits carrying high frequency voltages and currents cause electromagnetic interference (EMI) in them as well as in neighboring circuits. This EMI takes the form of induced voltages and currents due to electromagnetic coupling and electrostatic coupling between circuits as well as due to electromagnetic radiation. Every circuit carrying time-varying voltage and current acts as a transmitting antenna and receiving antenna simultaneously. The induced voltages and currents can lead to malfunction in circuits if not actual damage.

EMI happens at all frequencies. However, the induced voltages are usually of negligible magnitude at low frequencies. Therefore, a designer will often be forced to take out high frequency content from circuit waveforms for reducing destructive electromagnetic interference. An appreciation of how the Fourier series coefficients vary with harmonic order and the factors governing such variation helps him in such a task.

We noted in Example : 13.6-1 and subsequent examples in Section 13.6 that periodic impulse train waveforms of different type will have Fourier series with coefficients which *do not vary* with harmonic order n. The amplitude of harmonics is independent of harmonic order in such waveforms.

Integration brings a factor of $1/n$ in the Fourier series coefficients. Integration of an impulse train results in a waveform that will have step discontinuities in one period. Then *Integration in time* property of Fourier series shows us that such waveforms which contain step discontinuities will have Fourier series coefficients $\propto 1/n$ where n is the harmonic order.

> If a periodic waveform $v(t)$ requires m successive differentiation operations before impulses make their appearance, then, the harmonic amplitude in its Fourier series will decrease with $1/n^m$ at the least.

Integrating a square or a rectangular pulse waveform result in a piece-wise linear waveform. Such waveforms are continuous; but their first derivative will have step discontinuities. Their second derivative will contain impulse train along with other possible components. Thus we see that periodic waveforms that have impulses in their second first derivative will have Fourier series with coefficients decreasing with $1/n^2$ at the least. There may be terms involving $1/n^3$, $1/n^4$ etc. in the Fourier series of such a waveform; but the terms involving $1/n^2$ are the ones which decide how many terms in the Fourier series are to be included in a circuit analysis context or EMI context.

By extending this reasoning we may state qualitatively that if a periodic waveform $v(t)$ requires m successive differentiation operations before impulses make their appearance, then, the harmonic amplitude in its Fourier series will decrease with $1/n^m$ at the least.

Example : 13.8-1

Some desktop UPS units supplying single PC units deliver the waveform $v(t)$ shown in Fig. 13.8-1. The trapezoidal shape is expected to reduce the THD of the wave compared to the square wave in Example : 13.7-1. (a) Obtain an expression for r.m.s value of this voltage in terms of V and α. (b) Find the Fourier series coefficients for $v(t)$ (c) Find V and α such that the third harmonic content is zero and the r.m.s value is 220 V. (d) With these values for V and α find the THD of $v(t)$.

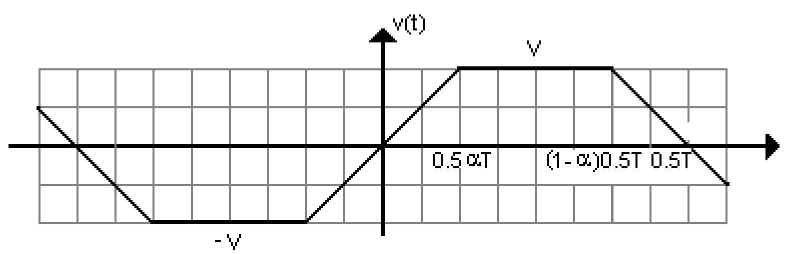

Fig. 13.8-1 Waveform for Example : 13.8-1

Solution

The waveform $v(t)$ exhibits *odd* symmetry and *half-wave* symmetry. Its Fourier series will contain only odd harmonics of sine format.

(a) r.m.s value of $v(t) = \sqrt{\dfrac{4}{T}\displaystyle\int_0^{0.5\alpha T}\left(\frac{2Vt}{\alpha T}\right)^2 dt + V^2(0.25T - 0.5\alpha T)} = V\sqrt{1-\tfrac{4}{3}\alpha}$

(b) Fourier series coefficients can be found by a straight application of the analysis equation of Fourier series Eqn. 13.2-2. However, a more elegant method will be to think of a waveform which will produce $v(t)$ on integration, find its Fourier series coefficients and obtain the Fourier series coefficients of $v(t)$ by dividing those coefficients by $jn\omega_o$. The waveform shown in Fig. 13.8-2 is the required one.

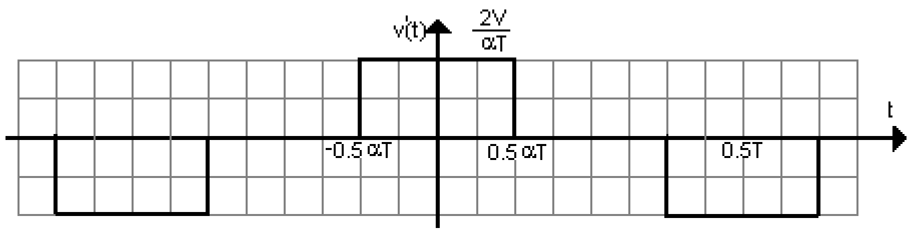

Fig. 13.8-2 Waveform of Derivative of $v(t)$ in Example : 13.8-1

Exponential Fourier series coefficients for this waveform is found by splitting the waveform into two – the upper half-cycles alone will constitute a periodic rectangular pulse train with period T and the lower half-cycles alone will constitute the same rectangular pulse train delayed by $0.5T$ and negated. Refer Example : 13.6-8 for Fourier series of such a periodic rectangular pulse train.

$$\tilde{v}_n' = \frac{2V}{\alpha T}\frac{1}{jn\omega_o T}\left(e^{j0.5n\omega_o\alpha T} - e^{-j0.5n\omega_o\alpha T}\right) - \frac{2V}{\alpha T}\frac{e^{-j0.5n\omega_o T}}{jn\omega_o T}\left(e^{j0.5n\omega_o\alpha T} - e^{-j0.5n\omega_o\alpha T}\right)$$

$e^{-j0.5n\omega_o T} = e^{-jn\pi} = 1$ for even n and -1 for odd n

$$\therefore \tilde{v}_n' = \begin{cases} 0 \text{ for even } n \\ \dfrac{2V}{\alpha T}\dfrac{2}{jn\omega_o T}\left(e^{j0.5n\omega_o\alpha T} - e^{-j0.5n\omega_o\alpha T}\right) \text{ for odd } n \end{cases}$$

$$= \begin{cases} 0 \text{ for even } n \\ \dfrac{4V}{T}\dfrac{\sin \pi n\alpha}{\pi n\alpha} \text{ for odd } n \end{cases}$$

Coefficients of exponential Fourier series of $v(t)$ is obtained by dividing these coefficients by $jn\omega_o$.

$$\therefore \tilde{v}_n = \begin{cases} 0 \text{ for even } n \\ -j\dfrac{2V}{\pi n}\dfrac{\sin \pi n\alpha}{\pi n\alpha} \text{ for odd } n \end{cases} \qquad (13.8\text{-}1)$$

The rate of decay of harmonic amplitude must be $\propto \frac{1}{n^2}$ and it is seen to be so. Notice that the waveform becomes a square waveform when $\alpha = 0$. The Fourier series coefficients given in Eqn. 13.8-1 become the same as that of a square wave *i.e.*, $\frac{2V}{\pi n}$ for odd n And when $\alpha = 0.5$, $v(t)$ becomes a triangle waveform with $\frac{4V}{\pi^2 n^2}$ for odd n as magnitude of exponential Fourier series coefficient magnitude.

(c) Third harmonic content will go to zero when $\sin 3\pi\alpha = 0$. Therefore, $\alpha = 1/3$ is the required value. The r.m.s value with this value of α is 0.7453 V and for 220 volts r.m.s value V must be ≈ 295 volts.

(d) The THD is evaluated as

$$THD = 100 \times \sqrt{\sum_{\substack{n=3 \\ odd\ n}}^{\infty} \frac{1}{n^2}\left(\frac{\sin n\pi/3}{n\pi/3}\right)^2} \div \frac{\sin \pi/3}{\pi/3} \approx 4.6\% .$$

The faster decay of harmonic amplitude with harmonic order due to the slanting portion of this trapezoidal waveform has yielded a much better approximation to a pure sine wave than the waveform in Example : 13.7-1. The next step to improve THD further will be to replace the flat portion of the waveform by another slanting portion with lesser slope than in the first section.