Introduction

Impulse response of a dynamic circuit completely characterizes it. The zero-state response for any arbitrary input can be obtained by convolving the input function with the impulse response in the time-domain. Non-zero initial conditions, if present, can also be handled by replacing them by impulse sources. Only that impulse response relevant to the corresponding point of application of source should be used.

We found that the sinusoidal steady-state frequency response function of a *stable* circuit is intimately related to its impulse response. Thus the frequency response function contains the impulse response in a disguised manner. This prompted us to ask whether we can work out the zero-state response of a *stable* circuit to an arbitrary input from its frequency response function. In short, does frequency response function of a *stable* circuit characterize it completely?

The last chapter provided a partial answer to this question. We considered the application of periodic inputs to a circuit in that chapter. We found that a well-behaved periodic input can be *resolved* into infinitely many pure sinusoidal components with their frequencies related harmonically. We were able to obtain the steady-state response to such inputs by using superposition principle and frequency response function. A waveform is periodic only if it exists for all t from $-\infty$ to $+\infty$. There is no natural response component in the output when such a periodic input is applied to a circuit. But that is not the way we drive circuits in practice. We switch on a periodic waveform at some time instant (usually at $t = 0$) to the circuit. Then the response will contain a transient part and forced response (or steady-state response) part. The method of analysis we discussed in the last chapter gives the steady-state response component in this case. It is so because forced response is the particular integral component of the solution of the differential equation governing the circuit and particular integral is the solution obtained by assuming that the input was applied from $-\infty$ onwards.

Notice that when we switch on a periodic input to a circuit, the resulting applied input *is not* a periodic function. For example $v_S(t) = \sin(\omega t)$ is periodic; but $v_S(t) = \sin(\omega t) \times u(t)$ is not periodic. And that is why there is a transient response component.

An aperiodic input is an input in which no periodicity can be identified. It may be of finite duration in time – *i.e.*, it may be identically zero in the entire time-domain except in a finite interval. Or it may be semi-infinite *i.e.*, it may be non-zero in $[0^+, \infty)$. It may have finite *normalised* energy or infinite *normalised* energy. We take up the issue of obtaining the complete response of a circuit to such aperiodic inputs in this chapter.

> This Chapter addresses the following questions.
>
> (i) Can an aperiodic waveform be resolved into sum of sinusoids of different amplitudes and frequencies? If yes, which are the frequencies that are present in the signal decomposition?
>
> (ii) Can steady-state frequency response function help us to get the output when such an aperiodic waveform is applied to the circuit? If yes, how do we explain a *steady-state* term yielding a *transient response* term?
>
> The first question will lead us to an important analytical tool in the study of signals and systems – *Fourier Transforms*.
>
> The second question will lead us to an important concept in study of linear systems – the *System Function*.
>
> The answers we develop for these two questions will reveal to us that *sinusoidal steady-state frequency response function* is a full and complete characterization of a linear, time-invariant (LTI) stable circuit.
>
> We will also derive considerable insight into time-domain versus frequency-domain behaviour of signals and circuits using Fourier Transforms.

14.1 Aperiodic Waveforms

All inputs get switched on to circuit at some time instant and get switched off or removed from the circuit at some other time instant in practice. Hence, all waveforms applied to circuits are *aperiodic* in practice. However, in steady-state studies, we may assume that the input is applied at $t = 0$ but not removed later. In that case, we have an aperiodic waveform that is *semi-infinite* in time. A periodic waveform switched on to the circuit at $t = 0$ is an example of an aperiodic input that is semi-infinite in time-domain. Such an aperiodic input has finite average power over a cycle, infinite duration and infinite energy – power and energy in the context of waveforms are to be understood as *normalised power and normalised energy.*

A periodic waveform switched on to a circuit at $t = 0$ and switched off later at $t = t_0$ is an example of an aperiodic input with finite duration and finite energy. The underlying waveform in these two examples was assumed to be periodic. But that assumption is necessary. The time-function describing a semi-infinite or finite duration signal can be any well-behaved function.

Aperiodic waveforms are classified in terms of their duration, energy and power. They can be *semi-infinite* or *finite duration* in time. They may have *finite energy* or *finite power*.

An aperiodic waveform that is finite in value and is of finite duration will have finite energy too. An aperiodic waveform of semi-infinite duration can have finite energy or finite power. Some examples follow.

An aperiodic waveform described by $v_S(t) = [u(t-\tau/2) - u(t+\tau/2)]$ is a *finite-duration , finite-energy* waveform. It is a rectangular pulse of unit height located in the interval $(-\tau/2, \tau/2)$. Its duration is τ sec and its normalised energy is τ Joules. The unit of normalised energy is Joules if it is understood as the energy that will be dissipated in a 1-Ohm resistance if this signal is either the voltage across it or the current through it. However, if normalised energy is understood as the value of the integral of $[v_S(t)]^2$ over the entire time-domain, then its unit will be volt^2-sec. Both units are used in practice. We use *Joules*.

Examples for different kinds of aperiodic signals.

An aperiodic waveform described by $v_S(t) = e^{-\alpha t}\, u(t)$ with a positive real value for α is a *semi-infinite duration, finite-energy* signal. Its energy content is $(0.5/\alpha)$ Joules.

An aperiodic waveform described by $v_S(t) = u(t)$ *i.e.*, the unit step function , is a *semi-infinite duration, infinite-energy* signal. However it has a *finite-power* of 1 W for $t > 0$.

An aperiodic waveform described by $v_S(t) = \sin(\omega t)\, u(t)$ is a *semi-infinite duration, infinite-energy* signal. Its average power content is 0.5 W for $t \geq (2\pi/\omega)$. The average power content is calculated over one period.

An aperiodic waveform described by $v_S(t) = \sin(\omega t)\, [u(t)-u(t-t_0)]$ is a *finite duration, finite-energy* signal. Its energy content is $0.5\omega t_0\, [1-(\sin 2\omega t_0/2\omega t_0)]$ Joules.

Finite-duration aperiodic signal as one period of a periodic waveform

It is not that aperiodic waveforms can not be expressed as a Fourier series. At least finite-duration aperiodic waveforms can be expressed as one period of a periodic waveform that has a Fourier series.

Let $v(t)$ be an aperiodic waveform with finite duration and let its duration be τ sec. Then we construct a periodic waveform $v_{\sim}(t)$ with a period of T sec ($T \geq \tau$) by placing the waveshape of $v(t)$ in every interval of width T sec from $-\infty$ to $+\infty$. Clearly, every period of $v_{\sim}(t)$ will be identical to $v(t)$. This periodic $v_{\sim}(t)$ will have a Fourier series, *i.e.*,

$$v_{\sim}(t) = \sum_{n=-\infty}^{\infty} \tilde{v}_n\, e^{-j\omega t}$$

where $\tilde{v}_n$ represents the n^{th} coefficient of the exponential Fourier series of $v_{\sim}(t)$. Now, we may write the aperiodic waveform as $v(t) = v_{\sim}(t) \times g(t)$ where $g(t)$ is a gate function defined as 1 for $0 < t < T$ and zero elsewhere. Fig. 14.1-1 shows this operation for an example triangular pulse aperiodic waveform. Here $\tau = 4$ s and $T = 5$ s.

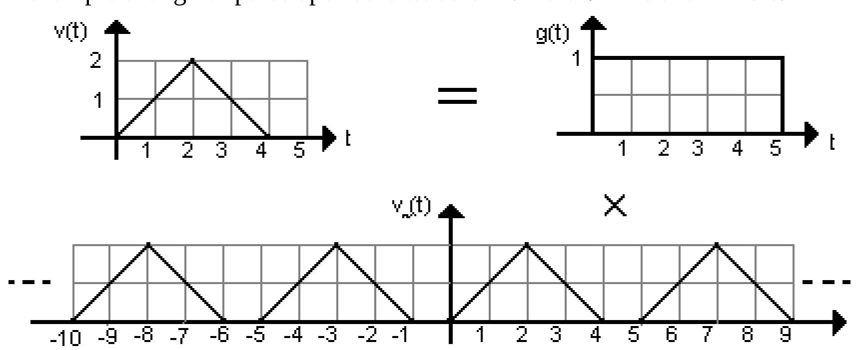


Fig. 14.1-1 A finite-duration aperiodic waveform as one period of a periodic waveform

But, can we obtain the *total response* of a circuit when $v(t)$ is applied as input from the output of the same circuit when $v_{\sim}(t)$ is applied to it? Essentially we are asking another equivalent question here – if we know the output $r_1(t)$ of an LTI circuit for an input $v_1(t)$, can we find out the output of the same circuit when $v_2(t)\, v_1(t)$ is applied to it by multiplying $r_1(t)$ by $v_2(t)$? The answer, obviously, is *no* since this is not *scaling* an input by a constant and hence principle of homogeneity does not apply. Similarly if $r_1(t)$ is the output when $v_1(t)$ is applied and $r_2(t)$ is the output when $v_2(t)$ is applied, $r_1(t)\ r_2(t)$ *is not* the output when $v_1(t)\ v_2(t)$ is applied to the circuit. Superposition principle covers only scaling by a constant and linear combinations of inputs. Multiplication of two functions of time is not a linear operation - *i.e.,* it does not obey superposition principle.

> A finite-duration aperiodic waveform can be expressed as one period of a periodic waveform that has a Fourier series.
>
> But this does not help us in solving for the circuit output in a convenient manner.
>
> However, this point of view will lead us to the Fourier representation of aperiodic waveforms as explained Section 14.2.

14.2 Fourier Transform of an Aperiodic Waveform

We could not work out the output of the circuit with an aperiodic input $v(t)$ from the output corresponding to its periodic version $v_\sim(t)$ since there are infinitely many periods in $v_\sim(t)$ and $v(t)$ is only one period of it. We can expect the responses to be the same only if $v(t)$ and $v_\sim(t)$ are identical waveforms. But that is possible if and only if $v_\sim(t)$ contains just one period because $v(t)$ is one period extracted from $v_\sim(t)$. A periodic waveform can have only one period in it only if its period spans the entire time-domain from $-\infty$ to $+\infty$. Therefore we want $v(t)$ to be $\lim_{T\to\infty} v_\sim(t)$. This limit has to be understood as the limit when the period extends to ∞ on both sides of the time-axis. And we expect to obtain the output of the circuit when $v(t)$ is applied by obtaining the output when the Fourier series of $v_\sim(t)$ is applied to the circuit with the condition that the period of the waveform $v_\sim(t)$ spans the entire time-axis.

Fourier Transform of a Finite-Duration Aperiodic Waveform

But what happens to the Fourier series of a periodic waveform when its period extends to ∞ on both sides in time-axis? We consider a specific example for this purpose. It is a rectangular pulse of width τ sec and height $1/\tau$ enclosing an area of unity.

The choice of zero time instant with respect to the waveform of a periodic signal affects only the phase spectrum of the signal. It does not affect the amplitude spectrum. Hence, for convenience and with no loss in generality, we assume that the rectangular pulse is located between $-\tau/2$ and $+\tau/2$ in the time-axis. The rectangular pulse as well as its periodic extension is shown in Fig. 14.2-1.

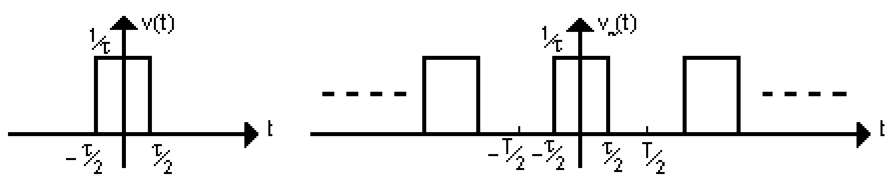


Fig. 14.2-1 A Rectangular Pulse and its Periodic Extension

The exponential Fourier series of $v_\sim(t)$ is found as

Let $\Delta\omega = \frac{2\pi}{T}$ rad/sec, i.e., the fundamental angular frequency of $v_\sim(t)$

$$\text{Then } \tilde{v}_n = \frac{1}{T}\int_{-T/2}^{T/2} v_\sim(t)\, e^{-jn\Delta\omega t}\, dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} \frac{1}{\tau} e^{-jn\Delta\omega t}\, dt$$

$$= \frac{-1}{jn\Delta\omega T\tau}\left[e^{-jn\Delta\omega\tau/2} - e^{jn\Delta\omega\tau/2}\right]$$

$$e^{-jn\Delta\omega\tau/2} - e^{jn\Delta\omega\tau/2} = -2j\sin\frac{n\Delta\omega\tau}{2} \text{ by Euler's Formula} \qquad (14.2\text{-}1)$$

$$\therefore \tilde{v}_n = \frac{2\sin\frac{n\Delta\omega\tau}{2}}{n\Delta\omega T\tau} = \frac{1}{T}\frac{\sin\frac{n\Delta\omega\tau}{2}}{\frac{n\Delta\omega\tau}{2}} = \left(\frac{1}{T}\right)\left(\frac{\sin x}{x}\right) \text{ where } x = \frac{n\Delta\omega\tau}{2}$$

For a special case of $\tau = 1$sec ,

$$\tilde{v}_n = \left(\frac{1}{T}\right)\left(\frac{\sin x}{x}\right) \text{ where } x = \frac{n\Delta\omega}{2} = \frac{n\pi}{T}$$

We observe the following in connection with this Fourier series.

(i) The exponential Fourier series coefficients are real-valued. There are only two possible values of phase – 0 corresponding to a positive real number and π rad corresponding to a negative real number. Therefore we can combine the amplitude and

> **The *sinc* function is defined as sinc(x) = sin(πx)/πx.**
>
> **$\therefore$ sin(x)/x = sinc(x/π)**

phase spectra into a single spectrum which will show both positive and negative components.

(ii) All frequency components are scaled by $1/T$. Therefore the amplitude of spectral components will decrease with the period.

(iii) The variation of amplitude of spectral components with the harmonic order, *i.e.*, with the value of frequency, is given by a sin x/x function. This function appears very frequently in study of signals and systems and is given a special name – the sinc(x/π) function.

The two-sided spectrum of this Fourier series is shown in Fig. 14.2-2 for three different values of T = 2.5 s, 5 s and 10 s. The sinc function shape is also shown as the envelope of spectral lines. Angular frequency values are used in the horizontal axis. sinc(x/π) goes to zero for the first time on two sides of origin when $x = \pm\,\pi$. This happens when $n\Delta\omega = 2\pi$. There may not be a spectral component at that point since n may not be an integer when $n\Delta\omega = 2\pi$. However, if we treat $\omega = n\Delta\omega$ as a continuous variable and use it in the horizontal axis, the envelope of the spectral lines will cross the axis when $\omega = 2\pi$ rad/sec.

> Fig. 14.2-2 shows the spectral components of a rectangular pulse of unit height and unit width subjected to infinite periodic extension.
>
> The spectral components are shown for three values of period of extension.
>
> Observe that when the period of extension is increased the number of spectral components appearing in any given band of frequencies increase. This crowding of spectral components is accompanied by a reduction in their amplitude.
>
> When we observe one of the two related quantities increasing while the other decreases, we are tempted to ask whether their product remains constant.
>
> We ask that question in this context and the effort to answer it leads us to the notions of 'spectral amplitude density' and Fourier transform.

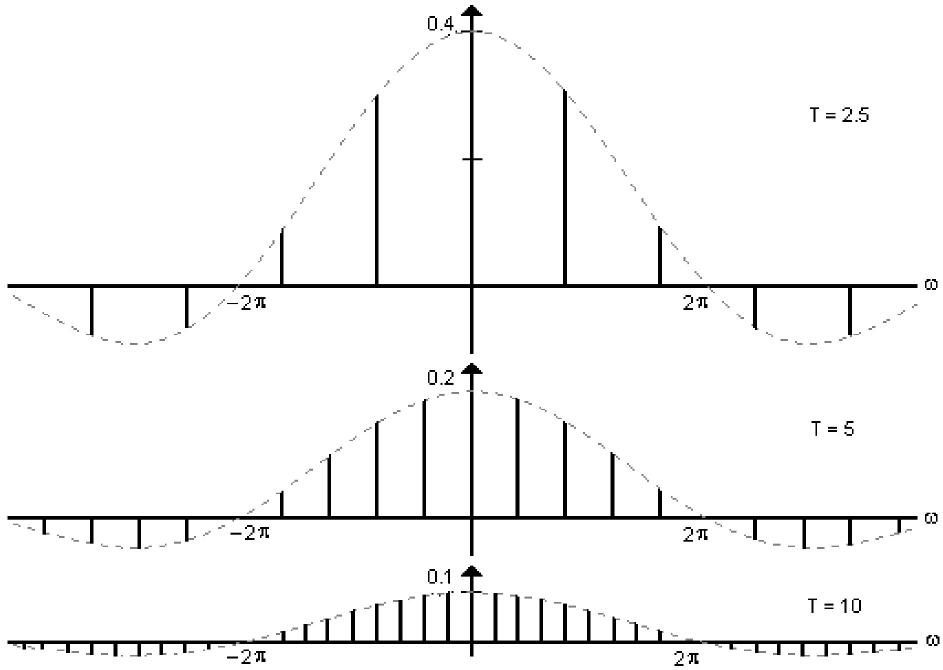


Fig. 14.2-2 Spectral Plots of Periodic Extension of a Rectangular Pulse of Unit Area and Unit Width for Various Periods of Extension

Consider a fixed angular frequency interval [-2π, 2π]. The number of spectral components which appear within this frequency range increase with T. That is, more and more frequency components appear within a given band of frequencies as we increase the period of extension. However, the amplitude of spectral components decreases when we do so. And the separation between the spectral components - $\Delta\omega = 2\pi/T$, the fundamental angular frequency – comes down as we increase T. This is true about any frequency range that we may consider. The envelope of spectral components always has a sinc(x/π) shape , but gets reduced in height as we increase T.

As $T \rightarrow \infty$, the number of spectral components in any given band of frequencies approach infinity with the amplitude of each component approaching zero, *i.e.*, infinitesimal. And the spectral components become so close that there is a spectral component at every real value of ω. The spectral envelope curve, while retaining its shape, gets reduced to infinitesimal in height.

But a spectrum with infinitesimal amplitudes is not good to look at! Hence we try the following artifice.

For a special case of $\tau = 1\text{sec}$,

$$\tilde{v}_n = \left(\frac{1}{T}\right)\left(\frac{\sin x}{x}\right) \text{ where } x = {}^{n\Delta\omega}\!/_{2} = {}^{n\pi}\!/_{T}$$

$$i.e., \tilde{v}_n T = \left(\frac{\sin x}{x}\right) \text{ where } x = {}^{n\Delta\omega}\!/_{2} = {}^{n\pi}\!/_{T} \quad (14.2\text{-}2)$$

$$i.e., \tilde{v}_n \frac{2\pi}{\Delta\omega} = \left(\frac{\sin x}{x}\right) \text{ where } x = {}^{n\Delta\omega}\!/_{2} = {}^{n\pi}\!/_{T}$$

Now we plot the quantity $\tilde{v}_n\, {}^{2\pi}\!/_{\Delta\omega}$ against ω. We divide the amplitude of spectral component by the frequency separation between adjacent spectral components and then scale it by 2π. It looks as if we are calculating the *average spectral amplitude density* available in a band of $\Delta\omega$ located at ω in doing so. We plot this quantity - 2π times *average spectral amplitude density* available in a band of $\Delta\omega$ located at ω against ω in Fig. 14.2-3.

Average spectral amplitude density introduced

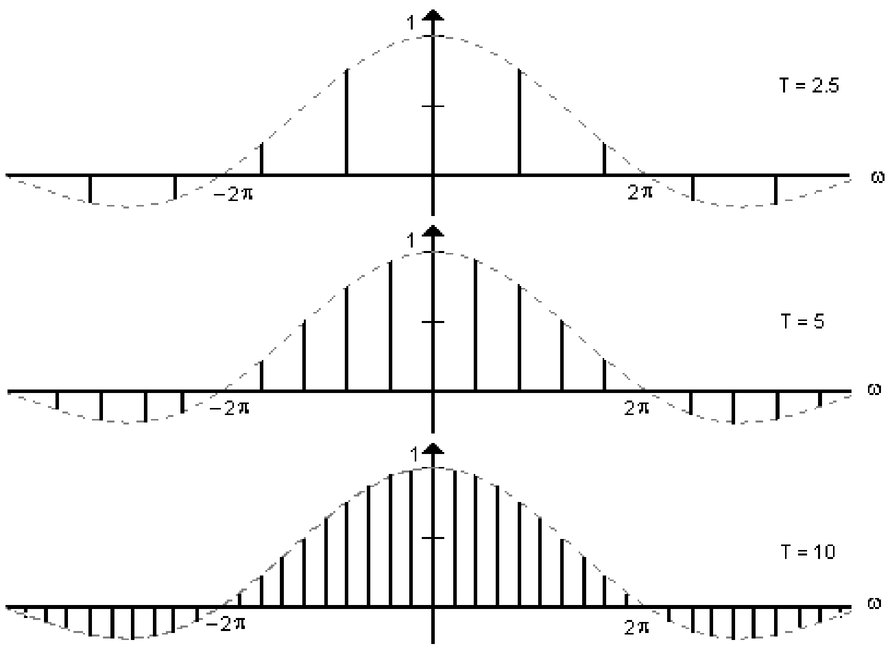


Fig. 14.2-3 Plot of Average Spectral Amplitude Density of a Rectangular Pulse Extension with Different Extension Periods

We get the same envelope in all the three cases. Now the effect of increasing T is to bring more and more spectral components into the same frequency band. *The fact that the amplitude of spectral components go down as we increase T is now hidden in the quantity that we plot – it is the spectral amplitude multiplied by T that we plot.*

Now, as $T \to \infty$, the quantity $\Delta\omega \to 0$, the quantity $n\Delta\omega \to \omega$ and there is a spectral component of infinitesimal amplitude at every imaginable value of ω. Further, though the spectral component amplitude at any ω is infinitesimal, the quantity $\tilde{v}_n\, {}^{2\pi}\!/_{\Delta\omega}$, which is a ratio of two infinitesimal quantities, approaches a finite limit. This limiting value at a particular value of ω is termed as *the value of Fourier transform of v(t) at ω.*

Meaning of Fourier transform value at a particular ω value

What we have illustrated in the case of a rectangular pulse applies to any finite-valued finite-duration aperiodic waveform. We formalize the above process for a finite-valued finite-duration aperiodic waveform and define Fourier transform as follows.

Let $\Delta\omega = \frac{2\pi}{T}$ rad/sec, i.e., the fundamental angular frequency of $v_{\sim}(t)$

Then $\tilde{v}_n = \frac{1}{T}\int_{-T/2}^{T/2} v_{\sim}(t)\, e^{-jn\Delta\omega t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} v(t)\, e^{-jn\Delta\omega t}\, dt$ since $v(t)$ is the same as $v_{\sim}(t)$ in this period.

$$\therefore \tilde{v}_n T = \tilde{v}_n \frac{2\pi}{\Delta\omega} = \int_{-T/2}^{T/2} v(t)\, e^{-jn\Delta\omega t}\, dt$$

As $T \to \infty, \Delta\omega \to 0$ and the discrete variable $n\Delta\omega \to \omega$ the continuous variable and $\tilde{v}_n \frac{2\pi}{\Delta\omega}$ goes to a limiting value.

$$\therefore \lim_{\Delta\omega \to 0} \tilde{v}_n \frac{2\pi}{\Delta\omega} = \int_{-\infty}^{\infty} v(t)\, e^{-j\omega t}\, dt$$

The limit on the left side is by definition the Fourier Transform $V(j\omega)$ of $v(t)$.

$$\therefore V(j\omega) = \int_{-\infty}^{\infty} v(t)\, e^{-j\omega t}\, dt$$

The time-function can be synthesized from the Fourier transform as shown below.

$$V(j\omega) = \lim_{\Delta\omega \to 0} \tilde{v}_n \frac{2\pi}{\Delta\omega} = \text{The limiting value of } \textit{spectral amplitude density} \times 2\pi$$

$\therefore$ The sum of spectral amplitude of all spectral components located at ω and in an infinitesimal band of $d\omega$ around $\omega = \frac{V(j\omega)}{2\pi} d\omega$

The time-function contributed by these spectral components $= \frac{V(j\omega)}{2\pi} e^{j\omega t} d\omega$

$v(t)$ = The sum of all such time-function contributions collected from the entire frequency-domain ($-\infty$ to $+\infty$)

$$\therefore v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)\, e^{j\omega t} d\omega$$

<u>Fourier transform pair – Eqn. 14.2-3</u>

The first equation describes the decomposition of an aperiodic waveform into infinite number of sinusoids, each with infinitesimal amplitude, and is called the *analysis equation* of Fourier Transform pair.

The second equation describes the construction of a time-function from its spectral decomposition and is called the *synthesis equation* of the Fourier Transform pair.

The following two expressions summarize the forward and inverse Fourier transform process.

$$V(j\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t} dt \qquad \text{The Analysis Equation}$$
$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)e^{j\omega t} d\omega \qquad \text{The Synthesis Equation} \qquad (14.2\text{-}3)$$

Fourier Transform of Infinite-Duration Aperiodic Waveforms

Infinite periodic extension of a given aperiodic waveform $v(t)$ and subsequent limiting process of sending the period T of periodic extension to ∞ lie behind the definition of Fourier transform. If $v(t)$ is of infinite duration, the waveform in one period of its periodic extension will be different from $v(t)$ for any finite T. However, when $T \to \infty$, the waveform within one period of $v_{\sim}(t)$ will be identical to $v(t)$ since there is only one period now. Therefore the definition of Fourier transform as given in Eqn. 14.2-3 is valid for such aperiodic waveforms too – provided the Fourier transform converges.

Fourier transform is assured to exist and converge for all finite-valued finite-duration aperiodic waveforms (except for some pathological waveforms that never put in their appearance in Circuit Analysis). However, there are semi-infinite aperiodic waveforms that are routinely used in circuit analysis for which Fourier transform as defined in Eqn. 14.2-3 will not converge and hence is not admissible.

If Fourier transform exists and converges for a semi-infinite aperiodic waveform, then, the meaning and interpretation of Fourier transform is the same as in the case of finite-duration aperiodic waveforms.

Interpretation of Fourier Transforms

(a) A periodic waveform can be resolved into sum of infinite sinusoidal waveforms. An aperiodic waveform also can be resolved into sum of infinite sinusoidal waveforms. However, there is a difference between the two 'infinities'. The 'infinity' of number of spectral components in the decomposition of a periodic waveform is the 'infinity' of integers. The 'infinity' of number of spectral components in the decomposition of an aperiodic waveform is the 'infinity' of real numbers. The spectrum of the waveform that displays the amplitude and phase of individual sinusoidal components against angular frequency is a discrete one in the case of periodic waveform and is a continuous one in the case of aperiodic waveform.

Differences between spectral descriptions for a periodic waveform and an aperiodic waveform.

(b) Therefore, only sinusoids that have their frequencies as integer multiples of some basic frequency value are present in a periodic waveform. But sinusoids of *all* frequencies are present in an aperiodic waveform.

(c) Though sinusoids of all frequencies are present in an aperiodic waveform, the amplitude of sinusoidal component *at any particular frequency* is infinitesimally small. An infinite number of such sinusoids with infinitesimally small amplitudes reinforce each other or cancel each other at various instants to bring forth the waveshape of the aperiodic waveform. *Infinitesimally small quantities are not necessarily negligible when there are infinite number of them.*

(d) We introduced the concept of *signal space* in an earlier chapter. It is a complex plane in which each point will represent a complex exponential signal with a definite complex frequency. Pure sinusoids are represented by points on the vertical axis (the imaginary axis) – the $j\omega$-axis. Hence Fourier series of a periodic waveform is *an expansion of a time function along the $j\omega$-axis in the signal space.* Similarly, Fourier transform of an aperiodic waveform is also an expansion of a time function along the $j\omega$-axis in the signal space.

Fourier series of a periodic waveform and Fourier transform of an aperiodic waveform are *expansions of time-functions* along $j\omega$-axis in s-plane.

(e) Fourier transform has its roots in complex exponential Fourier series and hence it is a two-sided complex function of a real variable ω. That it is a complex function is denoted explicitly by including $j\omega$ as the independent variable in the symbol $V(j\omega)$ – the j indicates the complex nature of the function rather than of the independent variable.

(f) Fourier transform *does not* give the amplitude and phase of spectral components. It can not do that since the amplitude of spectral component at any particular frequency is an infinitesimal. Rather, it gives the *density* of spectral amplitudes. Consider a particular frequency ω rad/sec and a small band of frequencies around it denoted by $\Delta\omega$. Collect all the infinitesimal amplitudes of all the sinusoids in that band and add them up – remember that it is a complex addition. Since there are infinite components in $\Delta\omega$, the sum of infinitesimal amplitudes need not be an infinitesimal. Now divide the sum by $\Delta\omega$. We get a quantity that has the dimensions of volt per rad/sec (assuming an aperiodic voltage waveform). Repeat the process with a smaller $\Delta\omega$ around the same ω. We will observe that the ratio – which is the *density of spectral amplitudes* at and around ω - approaches a limit as we reduce $\Delta\omega$ to zero. Fourier transform value at ω is this *limiting density of spectral amplitudes* multiplied by 2π. Hence Fourier transform is a *spectral amplitude density function*. We may express $\Delta\omega$ as $2\pi\Delta f$ where f is the cyclic frequency. Then, Fourier transform value at any ω gives the density of spectral amplitude at that frequency in volts/Hz unit.

Fourier transform value at any ω gives the density of spectral amplitude at that frequency in volts/Hz unit.

(g) Therefore, though Fourier transform does not tell us anything about the amplitude of sinusoid at a particular frequency contained in our aperiodic waveform (except that it is infinitesimally small), it tells us that the sum of complex amplitudes of all sinusoids with frequencies in the band $[\omega_0\text{-}\Delta\omega/2\ ,\ \omega_0+\Delta\omega/2]$ is $\approx V(j\omega_0) \times \Delta\omega/2\pi$ provided $\Delta\omega$ is small.

(h) What is the time-function contributed by sum of all those sinusoidal components located in the frequency band $[\omega_0\text{-}\Delta\omega/2\ ,\ \omega_0+\Delta\omega/2]$? Each frequency component contributes a function of the form $e^{j\omega t}$ with amplitude that is infinitesimal. However, the value of ω is changing from $\omega_0\text{-}\Delta\omega/2$ to $\omega_0+\Delta\omega/2$ in the band. Hence, we are adding sinusoids with slightly different frequencies (we get real sinusoids when we take corresponding frequency points from the left and right sides of the spectrum). We saw in the last paragraph that the sum of complex amplitudes of all sinusoids with frequencies in the band $[\omega_0\text{-}\Delta\omega/2\ ,\ \omega_0+\Delta\omega/2]$ is $\approx V(j\omega_0) \times \Delta\omega/2\pi$. But complex amplitude includes phase information. How do we add phases of *different frequency* sinusoids? Strictly speaking, we can't.

"The sum of complex amplitudes of all sinusoids with frequencies in the band $[\omega_0\text{-}\Delta\omega/2,\ \omega_0+\Delta\omega/2]$ is $\approx V(j\omega_0) \times \Delta\omega/2\pi$ provided $\Delta\omega$ is small."

But this statement is to be interpreted carefully. The amplitudes involved in the sum are amplitudes of *different frequency* sinusoids!

Actually the contribution from the band $[\omega_0\text{-}\Delta\omega/2,\ \omega_0+\Delta\omega/2]$ to the aperiodic waveform is a *wave packet* resulting from interference of many sinusoidal waveforms with different frequencies rather than a steady sinusoid at ω_0 . Refer to Fig. 14.2-4 and related discussion.

(i) However, if $\Delta\omega$ is infinitesimally small, *i.e.*, if it is $d\omega$, we can assume that all the sinusoids are virtually at the same frequency and all the spectral components contribute $e^{j\omega_0 t}$. Then, the time-function contributed by sum of all those sinusoidal components located in the frequency band $[\omega_0\text{-}d\omega/2\ ,\ \omega_0+d\omega/2]$ will be $\approx [V(j\omega_0) \times d\omega/2\pi]\, e^{j\omega_0 t}$ and will appear nearly as a constant amplitude – single frequency sine wave. But the amplitude will be negligibly small.

(j) But if the width of the band is not so small, we have to divide $\Delta\omega$ into many small intervals, compute the time function contributed by those small intervals and *add the time-functions point-by-point.* Equivalently, we may evaluate the following integral.

Time-function contributed by all the spectral components in the angular frequency band $[\omega_o\text{-}\Delta\omega/2, \omega_o+\Delta\omega/2]$

$$= \frac{1}{2\pi}\left[\int_{-(\omega_0+\Delta\omega/2)}^{-(\omega_0-\Delta\omega/2)} V(j\omega)e^{j\omega t}dt + \int_{(\omega_0-\Delta\omega/2)}^{(\omega_0+\Delta\omega/2)} V(j\omega)e^{j\omega t}dt\right].$$

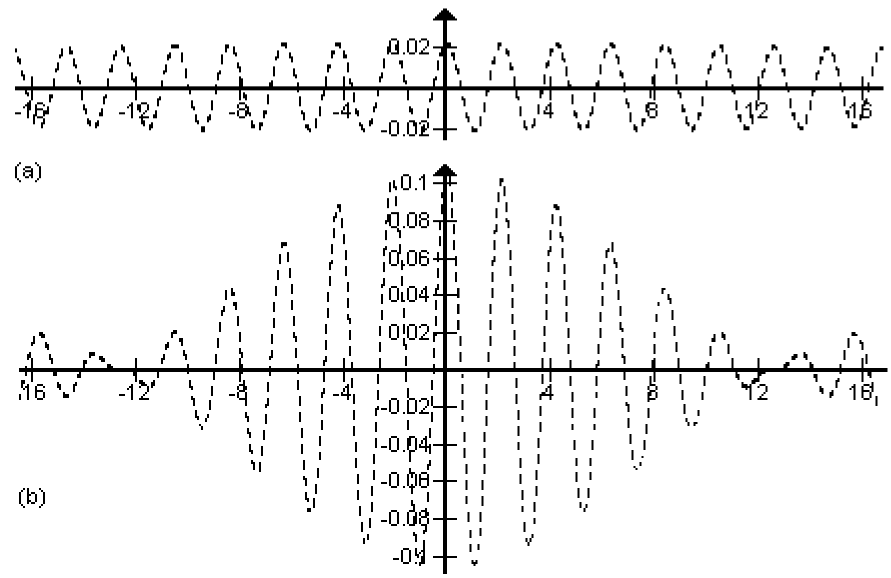


Fig. 14.2-4 Contribution of Frequency Band to a Rectangular Pulse of Unit Amplitude and Unit Width (a) [2.95, 3.05] rad/sec Band (b) [2.75, 3.25] rad/sec Band

Fourier transform was derived from Fourier series and hence each sinusoidal component in Fourier transform represents a periodic wave starting from -∞ and lasting up to +∞.

Infinite number of such sinusoids of different frequencies and infinitesimal amplitudes start from -∞ and proceed to evolve in time, interfering constructively at some instants and destructively at certain other instants in a manner suited to generate the desired aperiodic waveform.

(k) When two sinusoids of slightly different frequencies combine, they interfere with each other constructively for certain time intervals and destructively in certain other intervals. This results in the well-known beat phenomenon. This kind of constructive and destructive interference takes place to produce wave packets in the time-function contribution under discussion. This interference mechanism is clearly visible in (b) of Fig. 14.2-4. This figure shows the contribution of spectral components in the 2.75 rad/sec to 3.25 rad/sec band to the aperiodic waveform that is 1 V rectangular pulse located between –0.5 sec and +0.5 sec in the time-domain. The curve (a) in the same figure shows the contribution of spectral components in the 2.95 rad/sec to 3.05 rad/sec band to the same rectangular pulse. Since the spread of frequency is smaller in this case it will take more than 16 sec for the interference pattern to manifest. This waveform also will show waxing and waning when plotted for a larger duration. Also note that the peak amplitudes in the two cases are in the same ratio as that of the frequency bands.

14.3 Convergence of Fourier Transforms

The Fourier transform of an aperiodic waveform $v(t)$ is given by

$$V(j\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t}dt \qquad (14.3\text{-}1)$$

It may be possible to evaluate this integral for a given $v(t)$. But that does not make the integral the Fourier transform of $v(t)$. It becomes the Fourier transform of $v(t)$ only if $v(t)$ can be constructed *uniquely* from $V(j\omega)$ by means of the synthesis equation ,

$$v(t) \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)e^{j\omega t}d\omega \qquad (14.3\text{-}2)$$

Therefore, the integral in Eqn. 14.3-1 is the Fourier transform of $v(t)$ only if the integral in Eqn. 14.3-2 converges to the value of $v(t)$ for all t. If it does so, we assert that the Fourier transform exists for the signal $v(t)$.

"Convergence of Fourier transform" refers to the convergence of the *synthesis integral* and not to the convergence of the *analysis integral*. That is, we say a Fourier transform $V(j\omega)$ of a time-function $v(t)$ converges if the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)e^{j\omega t}\,d\omega$ converges to the value of $v(t)$ at the value of t used in the integral.

There is a set of conditions to be satisfied by a time-function for its Fourier transform to exist and converge. They are called the *Dirichlet's Conditions*. It is a set of *sufficient but not necessary* conditions for the Fourier transform to exist. This implies that even functions that violate one or more Dirichlet's conditions may have legitimate Fourier transforms. However, if a time-function satisfies all Dirichlet's conditions, the Fourier transform for it is guaranteed to exist and converge.

Dirichlet's conditions require that:

1. $v(t)$ be absolutely integrable , that is,

$$\int_{-\infty}^{\infty} |v(t)|dt < \infty \qquad (14.3\text{-}3)$$

2. $v(t)$ should have only a finite number of maxima and minima within any finite time interval

3. $v(t)$ should have only a finite number of discontinuities within any finite time interval. Also, these discontinuities must be finite.

If a function $v(t)$ satisfies these three conditions, its Fourier transform defined in Eqn. 14.3-1 exists and the synthesis integral in Eqn. 14.3-2 is assured to converge to the value of $v(t)$ for all *t except at a discontinuity*. At a discontinuity, the synthesis integral will converge to the average of the function before and after the discontinuity.

The second and third Dirichlet's conditions are usually met by all waveforms that have some practical application in circuits (except impulse function). Only some pathological waveforms manage to violate these conditions and we never meet with them in circuits. However, many commonly used waveforms violate the first Dirichlet's condition – that is, they are not absolutely integrable. However, this does not imply that they have no Fourier transform. Dirichlet's conditions are *sufficient conditions*; but not *necessary conditions.* Therefore, a function which violates one or more of them may still have a Fourier transform. Unit impulse function violates third Dirichlet's condition, but still has a Fourier transform. Unit step function violates first Dirichlet's condition, but still has a Fourier transform.

Dirichlet's conditions are *sufficient conditions*; but not *necessary conditions* for existence and convergence of Fourier transforms.

Dirichlet's conditions assure us that the synthesis integral in Eqn. 14.3-2 will converge to the average value of the function before and after the discontinuity at a point of discontinuity. Moreover, convergence to the actual function value is guaranteed at all other instants. But then, that will imply that the synthesis integral in Eqn. 14.3-2 is synthesizing a discontinuous time function if $v(t)$ is discontinuous at some time instant. We note that the synthesis integral is only a short form notation for sum of infinitely many sinusoids of infinitesimally small amplitudes covering a frequency range of 0 Hz to ∞ Hz. All those sinusoids are *continuous functions of time*. How can we get a discontinuous function by adding continuous functions even if we add infinite such functions?

The answer requires a more detailed look at the process of convergence of the synthesis integral. We take the example of a rectangular pulse for this purpose.

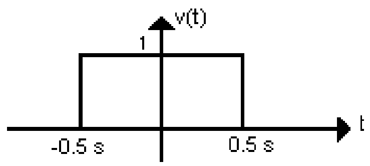


Fig. 14.3-1 Rectangular Pulses for Example : 14.3-1

Example : 14.3-1

The waveform of a rectangular pulse in time-domain is given in Fig. 14.3-1. (i) Obtain its Fourier transform and plot its continuous spectrum for a special case of unit amplitude and unit pulse-width. (ii) Invert its Fourier transform considering only those frequency components that fall below 50 rad/sec and plot the result. (iii) Invert its Fourier transform for the time interval [-0.52 s , -0.48 s] for (a) considering only those components which fall below 500 rad/sec and (b) considering only those components which fall below 2500 rad/sec and plot the result.

Solution

(i) The Fourier transform is obtained by applying the Eqn. 14.3-1 with $v(t) = 1/\tau$ in the interval $(-\tau/2\ ,\ \tau/2)$.

$$\begin{aligned}
V(j\omega) &= \int_{-\infty}^{\infty} v(t)e^{-j\omega t}dt = \int_{-\tau/2}^{\tau/2} \tfrac{1}{\tau}\, e^{-j\omega t}dt \\
\therefore V(j\omega) &= \frac{1}{-j\omega\tau}\left[e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right] \\
e^{-j\omega\tau/2} - e^{j\omega\tau/2} &= -2j\sin \omega\tau/2 \text{ by Euler's Formula} \\
\therefore V(j\omega) &= \frac{\sin \omega\tau/2}{\omega\tau/2} = \frac{\sin \omega/2}{\omega/2} \text{ when } \tau = 1\text{sec}
\end{aligned} \qquad (14.3\text{-}4)$$

The plot of the continuous spectrum is shown in Fig. 14.3-2. Since it is a real-valued spectrum both amplitude (*i.e.,* magnitude) and phase spectra are combined into one spectrum.

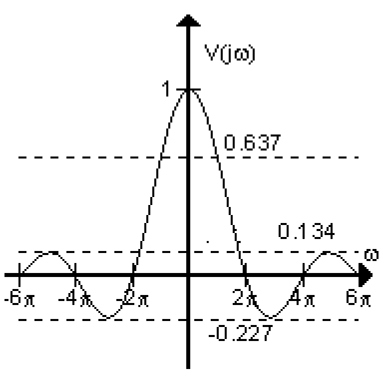


Fig. 14.3-2 Continuous Spectrum of a Rectangular Pulse of Unit Height and Unit Width

The spectral amplitude density function is found to vary as $\sin(\omega/2)/(\omega/2)$. Sinusoidal components at all frequencies except for $\omega = 2k\pi$ rad/sec, where k is a non-zero integer, are present in this rectangular pulse. However, the spectral amplitude density decreases in inverse proportion to angular frequency.

(ii) We evaluate the partial sum of infinite number of sinusoidal components which have their frequencies in the band 0 to 50 rad/sec to examine how closely will the result of partial synthesis $\hat{v}(t)$ match the original function $v(t)$. We remember that two frequency components at $+\omega$ and $-\omega$ from an exponential Fourier series or Fourier transform will contribute $e^{-j\omega t}$ and $e^{j\omega t}$ terms producing a sinusoidal wave. Therefore, when we accept a spectral component at ω we have to take its companion component located at $-\omega$ along with it. With this in mind, we write the partial synthesis as,

$$\hat{v}(t) = \frac{1}{2\pi}\int_{-50}^{50} V(j\omega)e^{j\omega t}d\omega$$

Since $V(j\omega)$ is an even real function in this case and

$e^{j\omega t} + e^{-j\omega t} = 2\cos\omega t$ by Euler's Formula,

$$\hat{v}(t) = \frac{1}{\pi}\int_{0}^{50} \frac{\sin(0.5\omega)}{(0.5\omega)}\cos\omega t\, d\omega$$

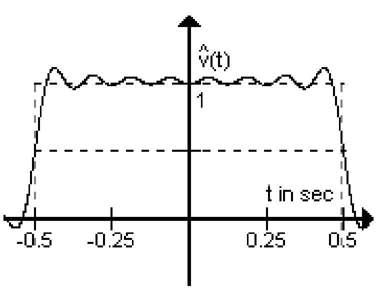


Fig. 14.3-3 Time-domain Waveform Synthesized from 0 to 50 rad/sec Spectral Components of the Pulse in Example : 14.3-1

This integral is evaluated numerically (a few lines of code in your favorite programming language will be enough) and the result is plotted in Fig. 14.3-3.

This waveform shows that the value at the discontinuity is close to the average value of 0.5. But the frequency components we included in the synthesis are evidently not enough to ensure convergence at all other points. In fact, a simulation study will show that the value of $\hat{v}(t)$ at any t will approach the value of $v(t)$ in an oscillatory manner as we increase the frequency of components included in the synthesis. As we make $\omega \to \infty$, the amplitude of these oscillations in value go down and the value of $\hat{v}(t)$ converges to value of $v(t)$ except at neighborhoods around points of discontinuity.

(iii) The oscillation in the synthesized waveform shown in Fig. 14.3-3 does not disappear as we increase the number of spectral components included in the synthesis. Rather, these oscillations get crowded towards the points of discontinuity. The result is that there is always a small neighborhood around the time instants of discontinuity where the difference between the synthesized waveform and original waveform will oscillate significantly. Increasing the range of frequency included in the synthesis will only reduce the width of this neighborhood, but will not succeed in reducing it to zero. Thus, there are some time points around points of discontinuity at which the synthesized wave will not converge to original wave – we can only reduce the range in time-domain where this happens by increasing the range of frequency included in the synthesis. Hence the synthesized waveform remains continuous even when $\omega \to \infty$ in the synthesis and can not match the discontinuous waveform we started with. There is always a small neighborhood of departure around points of discontinuity. This is illustrated by the two curves in Fig. 14.3-4 that show the results of synthesis using components in the 500 rad/sec and 2500 rad/sec ranges.

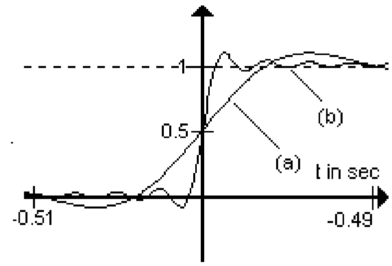


Fig. 14.3-4 Time-domain Waveform Synthesized from (a) 0 to 500 rad/sec and (b) 0 to 2500 rad/sec Spectral Components of the Rectangular Pulse in Example : 14.3-1

These oscillations of the difference between $\hat{v}(t)$ and $v(t)$ are called *Gibb's Oscillations* , named after Josiah Gibbs who explained them in 1899.

As $\omega \to \infty$ in the synthesis, these oscillations will get crowded so much that they will contain negligible energy and can be ignored safely.

In any case, we need not worry too much about them in circuit analysis. We will not be able to make physical waveforms containing discontinuities to begin with. We will see why later in this chapter. One has to generate such a discontinuous waveform before one applies it to some circuit – and one will not be able to make it!

Uniqueness of Fourier transform pair

Dirichlet's conditions are only *sufficient* conditions for existence of Fourier transform. How do we find out the Fourier transform for a function that violates one or more of these conditions? For all we know, it may not exist. One procedure would be to blindly use the analysis integral of Fourier transform pair and study the resulting Fourier transform for its convergence properties. Another method would be to express the troublesome function as a limiting case of some other function which satisfies all Dirichlet's conditions and try to arrive at the Fourier transform of the troublesome function as a limiting case of Fourier transform of this well-behaved function.

But, how do we know that the Fourier transform we have arrived at for a function that does not satisfy the Dirichlet's conditions by some means is indeed *the* Fourier transform of that function – or, in other words, can there be different Fourier transforms for same time-function and different time-functions for the same Fourier transform?

The theorem of Uniqueness of Fourier transform pair assures us that if we have found a Fourier transform function for a given time-function, then, that is the one and only Fourier transform that will exist for that function. Similarly, it assures that if we have found a time-function by inverting a Fourier transform, then, that is the one and only time-function with that Fourier transform. This theorem makes our efforts to identify Fourier transform of functions that do not satisfy Dirichlet's conditions worthwhile. Thus, the time-function $v(t)$ and its Fourier transform (if it exists) $V(j\omega)$ form a *unique pair* and this unique relationship between them is represented symbolically by $v(t) \Leftrightarrow V(j\omega)$.

14.4 Some Basic Properties of Fourier Transforms

Signals are subjected to various operations in the time-domain in electrical circuits and systems. Some of the basic signal operations are scaling by a constant, differentiation, integration, multiplication by another signal (this is called *time-domain modulation*), delaying or advancing in time etc. When a signal undergoes a time-domain operation its Fourier transform undergoes a corresponding operation in frequency-domain. We want to establish this correspondence through the study of properties of Fourier transforms. We cover some basic properties of Fourier transforms in this section and continue to develop more advanced properties that are relevant to circuit analysis in later sections. Simultaneously, we work out the Fourier transforms of many time-functions of importance to circuit analysis.

Linearity of Fourier Transform

Forward and Inverse Fourier Transformation involves mathematical integration in time-domain and frequency-domain respectively. Integration is a linear mathematical operation and obeys superposition principle. Therefore Fourier transforms obey superposition principle. That is,

If $v_1(t) \Leftrightarrow V_1(j\omega)$ and $v_2(t) \Leftrightarrow V_2(j\omega)$, then, $a_1v_1(t) + a_2v_2(t) \Leftrightarrow a_1V_1(j\omega) + a_2V_2(j\omega)$

Example : 14.4-1

Obtain the Fourier transform of the waveform in Fig. 14.4-1 and plot its continuous spectrum.

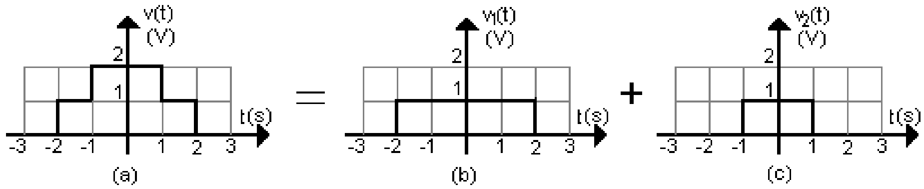


Fig. 14.4-1 Waveform for Example : 14.4-1 and its Decomposition

Solution

We have already worked out the Fourier transform of a symmetrically positioned rectangular pulse of unit area as (sin $\omega\tau/2)/(\omega\tau/2)$ where τ is the width of pulse and ω is angular frequency in rad/sec. If the height of rectangular pulse is V, its area will be $V\tau$ and by linearity property of Fourier transform its Fourier transform will be $(V\tau)$ (sin $\omega\tau/2)/(\omega\tau/2)$.The waveform $v(t)$ given in (a) of Fig. 14.4-1 can be decomposed into sum of $v_1(t)$ and $v_2(t)$ as shown in (b) and (c) of the same figure. Therefore the Fourier transform of $v(t)$ is the sum of Fourier transforms of $v_1(t)$ and $v_2(t)$. $\therefore V(j\omega) = 4\dfrac{\sin 2\omega}{2\omega} + 2\dfrac{\sin\omega}{\omega}$. This is plotted in Fig. 14.4-2.

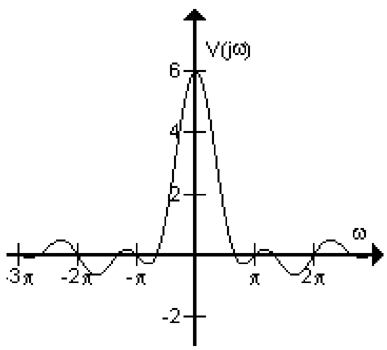


Fig. 14.4-2 Continuous Spectrum of Waveform in Example : 14.4-1

Duality in Fourier Transform

The analysis and synthesis equations of Fourier transform are repeated below. A close look at them reveals that except for a sign change in the exponent and a factor of 2π, the two integrals are similar. This similarity leads to an important property of Fourier transform – the property of *Duality*.

$$V(j\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t}dt \quad ---- \text{The Analysis Equation}$$

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)e^{j\omega t}d\omega \quad ---- \text{The Synthesis Equation}$$

We employ the following mathematical manipulation of variables to arrive at duality property.

$$V(j\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t}dt \quad ---- \text{The Analysis Equation}$$

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)e^{j\omega t}d\omega \quad ---- \text{The Synthesis Equation}$$

Let two new variables be defined as $t' = -\omega$ and $\omega' = t$.

Rewriting the analysis and synthesis equations in terms of t' and ω',

$$V(-jt') = \int_{-\infty}^{\infty} v(\omega')e^{j\omega' t'}d\omega' \text{ and}$$

$$v(\omega') = \frac{1}{2\pi}\int_{\infty}^{-\infty} V(-jt')e^{-j\omega' t'}(-dt') = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(-jt')e^{-j\omega' t'}dt'$$

Now we recast these equations as

$$[2\pi v(\omega')] = \int_{-\infty}^{\infty} V(-jt')\,e^{-j\omega' t'}dt'$$

$$V(-jt') = \int_{-\infty}^{\infty} [2\pi v(\omega')]\,e^{j\omega' t'}d\omega'$$

And now we get rid of the primed variables and bring back the original variables by substituting $\omega' = \omega$ and $t' = t$. We get,

$$[2\pi v(\omega)] = \int_{-\infty}^{\infty} V(-jt)\,e^{-j\omega t}dt \qquad \text{This is an } \textit{analysis equation.}$$

$$V(-jt) = \int_{-\infty}^{\infty} [2\pi v(\omega)]\,e^{j\omega t}d\omega \qquad \text{This is a } \textit{syntheis equation.}$$

Duality property of Fourier transforms

This shows that for every $v(t) \Leftrightarrow V(j\omega)$ pair there exists another Fourier transform pair $V(-jt) \Leftrightarrow 2\pi v(\omega)$. The $-j$ in $V(-jt)$ indicates that it is a complex function of a real variable t. The interpretation of this result is as follows.

Given a $v(t)$ with a waveshape in t-axis and its $V(j\omega)$ with a waveshape each for its real part and imaginary part (or magnitude and phase parts) in the ω-axis, visualize a new ω-axis and transfer 2π times the shape that $v(t)$ has in t-axis to this new ω-axis. Think of this transferred shape as a new Fourier transform of some time-function. If $v(t)$ was a real function of t, this Fourier transform has only real part and its imaginary part is zero. Now raise the question – which time-function has this Fourier transform?

The *Duality Property* answers – Visualize a new t-axis. Take the shape of $V(j\omega)$ in the original ω-axis (take the shape of its real and imaginary parts together) , reflect the shape on the vertical axis (*i.e.,* get the *mirror image* of the shape) and transfer the reflected shape to the new t-axis. You get a *complex function of a real variable (time)* now. That is the time-function you want.

Duality in Fourier transform

For every $v(t) \Leftrightarrow V(j\omega)$ pair there exists another Fourier transform pair $V(-jt) \Leftrightarrow 2\pi v(\omega)$.

This relationship is illustrated in the case of a rectangular pulse in Fig. 14.4-3. A rectangular pulse in time-domain has a frequency-domain description which has a *sinc function* shape. And by Duality property a rectangular pulse shaped Fourier transform has a time-domain waveform that has a *sinc function* shape.

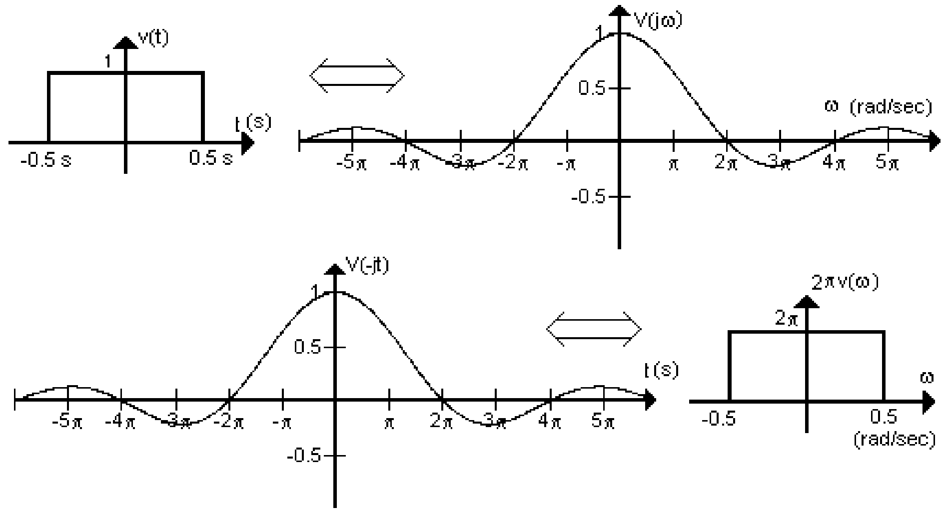


Fig. 14.4-3 Duality Property of Fourier Transforms Illustrated

Example : 14.4-2

Find a time-function $v'(t)$ such that its Fourier transform will have the same shape and magnitude as that of the time-function waveform shown in (a) of Fig. 14.4-4.

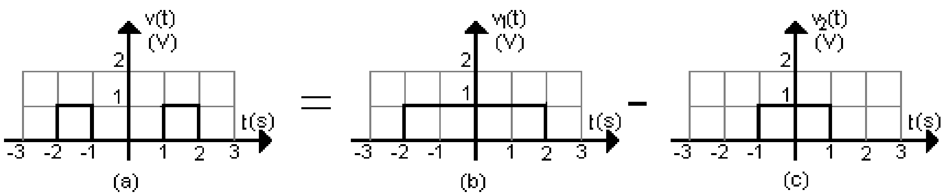


Fig. 14.4-4 Waveform of $v(t)$ for Example : 14.4-2 and its Decomposition

Solution

The waveform of $v(t)$ can be decomposed into $v_1(t)$ and $v_2(t)$ as shown in Fig. 14.4-4. $v(t)$ is $v_1(t) - v_2(t)$. The components $v_1(t)$ and $v_2(t)$ are the same as the components which appeared in Example : 14.4-1. Therefore Fourier transform of $v(t)$ will be $V(j\omega) = 4\frac{\sin 2\omega}{2\omega} - 2\frac{\sin \omega}{\omega}$ by using the property of linearity of Fourier transforms.

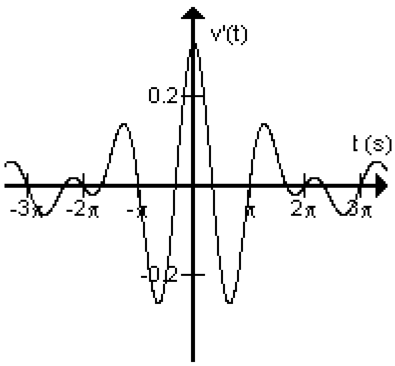


Fig. 14.4-5 Plot of the Required Time-Function in Example : 14.4-2

Now, by Duality Property, 2π times the waveshape in (a) of Fig. 14.4-4, if treated as a Fourier transform, will have an inverse of $4\frac{\sin 2(-t)}{2(-t)} - 2\frac{\sin(-t)}{(-t)} = 4\frac{\sin 2t}{2t} - 2\frac{\sin t}{t}$. Therefore the time-function which will have the waveshape (a) in Fig. 14.4-4 as its Fourier transform will be given by $v'(t) = \frac{1}{\pi}\left[2\frac{\sin 2t}{2t} - \frac{\sin t}{t}\right]$. Plot of this function is shown in Fig. 14.4-5.

We take note of a particular trend in the Fourier transform pair – we observe that when the function is sharply limited in one domain it gets extended from -∞ to +∞ in the other domain. A rectangular pulse in time-domain gets stretched out over the entire frequency-domain in the form of a *sinc* function. A function that has rectangular shape in frequency-domain gets spread out over the entire time-domain in the form of a *sinc* function. This tendency of functions to get stretched out in one domain when they are being confined in the other domain is a general tendency and is not specific to rectangular waveforms. It has many practical implications as far as electrical circuits are concerned. We will look at it in detail in a later section in this chapter.

Time Reversal Property

Assume that a signal $v(t)$ has a Fourier transform $V(j\omega)$. Let $v_1(t)$ be the time-reversed version of $v(t)$, *i.e.*, $v_1(t) = v(-t)$. Then the Fourier transform of $v_1(t) = V_1(j\omega) = V(-j\omega)$.This may be easily proved starting from the analysis equation. This implies that a reversal in time-domain is accompanied by a reversal in frequency-domain.

Time Shifting Property

Time-shifting property of Fourier transforms

Assume that a signal $v(t)$ has a Fourier transform $V(j\omega)$. Let $v_1(t)$ be the time-shifted version of $v(t)$, *i.e.*, $v_1(t) = v(t-t_d)$ where t_d is a constant time delay. Any value which is taken up by $v(t)$ at t is assumed by $v_1(t)$ at $t+t_d$ and thus the waveshape of $v_1(t)$ is same as the waveshape of $v(t)$ shifted forward, *i.e.*, delayed in time-axis. So when t_d is +ve the waveform gets delayed and when t_d is –ve the waveform gets advanced in time. We obtain an expression for the delayed version as below.

$$V_1(j\omega) = \int_{-\infty}^{\infty} v(t-t_d)e^{-j\omega t}dt. \text{ Substitute } t' = t - t_d. \text{ Then,}$$

$$V_1(j\omega) = e^{-j\omega t_d}\int_{-\infty}^{\infty} v(t')e^{-j\omega t'}dt' = e^{-j\omega t_d}V(j\omega)$$

The effect of delaying in time-domain is a multiplication by a factor of unit magnitude and a phase angle that varies linearly with frequency. Therefore, the amplitude of sinusoidal components contained in the signal does not change, but their phases get delayed further by an amount that is directly proportional to their frequency value. The constant of proportionality is t_d. Therefore, $v_1(t)$ will have same magnitude spectrum as that of $v(t)$ but its phase spectrum will have an additional linear phase delay contribution $-\omega t_d$ rad.

Example : 14.4-3

Find the Fourier transform of the waveform shown in (a) of Fig. 14.4-6 and plot its continuous magnitude and phase spectra.

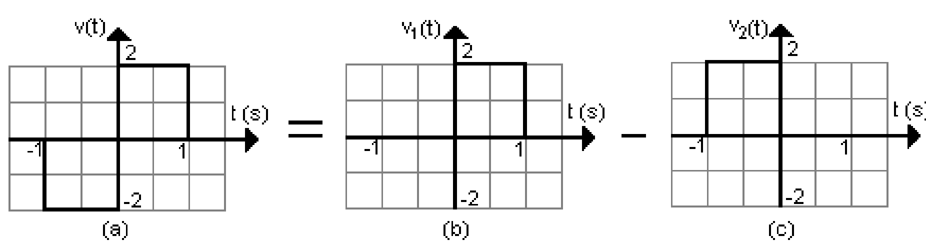


Fig. 14.4-6 Waveform for Example : 14.4-3 and its Decomposition

Solution

The decomposition of the given waveform is shown in (b) and (c) of Fig. 14.4-6. It can be seen that $v(t) = v_1(t) - v_2(t)$. But $v_2(t)$ is $v_1(t)$ reflected about the vertical axis, *i.e.*, $v_2(t) = v_1(-t)$. Therefore, $v(t) = v_1(t) - v_1(-t)$. We find $V_1(j\omega)$ first.

We notice that $v_1(t+0.5)$ will be a symmetrically placed rectangular pulse of height 2 and width 1 sec. It will be placed between –0.5 sec and 0.5 sec. We have already worked out the Fourier transform for such a pulse and it will be 2 $\sin(0.5\omega)/(0.5\omega)$. But this must be $e^{j0.5\omega} \times V_1(j\omega)$ by time shifting property of Fourier transforms.

$$\therefore V_1(j\omega) = \left[2\frac{\sin(0.5\omega)}{(0.5\omega)}\right]e^{-j0.5\omega}$$

By time-reversal property of Fourier transforms,

$$V_2(j\omega) = \left[2\frac{\sin(-0.5\omega)}{(-0.5\omega)}\right]e^{-j0.5(-\omega)} = \left[2\frac{\sin(0.5\omega)}{(0.5\omega)}\right]e^{j0.5\omega}$$

By Linearity property of Fourier transforms,

$$V(j\omega) = V_1(j\omega) - V_2(j\omega) = \left[2\frac{\sin(0.5\omega)}{(0.5\omega)}\right]\left[e^{-j0.5\omega} - e^{j0.5\omega}\right]$$

$$= \left[2\frac{\sin(0.5\omega)}{(0.5\omega)}\right]\left[-2j\sin(0.5\omega)\right] = -j4\left[\frac{\sin(0.5\omega)}{(0.5\omega)}\right]\sin(0.5\omega)$$

The Fourier transform is seen to be a pure imaginary one and a careless look at it may give us an impression that the phase spectrum is always at $-\pi/2$ rad. The *magnitude spectrum* of a Fourier transform is always positive valued since magnitude of a complex number is positive. Therefore, if the sign of $\sin^2(0.5\omega)/(0.5\omega)$ changes, then, there has to be an additional contribution of π radians for those values of frequency at which the sign is negative.

The magnitude and phase spectra are plotted in Fig. 14.4-7.

Continuous spectrum of a waveform can also be shown in terms of the real and imaginary parts of its Fourier transform. The real part of Fourier transform is zero in this example and the imaginary part is $-4\sin^2(0.5\omega)/(0.5\omega)$. The plot of imaginary part of Fourier transform is shown in Fig. 14.4-8.

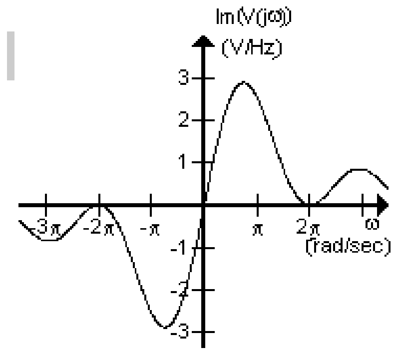


Fig. 14.4-8 Spectrum of Imaginary Part of Fourier Transform in Example : 14.4-3

Example : 14.4-4

Refer to Fig. 14.4-6. If instead of subtracting $v_2(t)$ from $v_1(t)$ it is added to $v_1(t)$ we get a rectangular pulse of height 2 and width 2 sec symmetrically placed between –1 sec and 1 sec. Find the Fourier transform of $v(t) = v_1(t) + v_2(t)$ and plot its spectrum.

Solution

The Fourier transforms of $v_1(t)$ and $v_2(t)$ have been derived in Example : 14.4-3. We get the Fourier transform of $v(t)$ by adding these two transforms.

$$V(j\omega) = V_1(j\omega) + V_2(j\omega) = \left[2\frac{\sin(0.5\omega)}{(0.5\omega)}\right]\left[e^{-j0.5\omega} + e^{j0.5\omega}\right]$$

$$= \left[2\frac{\sin(0.5\omega)}{(0.5\omega)}\right]\left[2\cos(0.5\omega)\right] = 4\left[\frac{\sin(0.5\omega)}{(0.5\omega)}\right]\cos(0.5\omega)$$

But we know that the Fourier transform of a rectangular pulse with a height of V and width τ is $(V\tau)\sin(\omega\tau/2)/(\omega\tau/2)$. It may easily be shown that this is the same as the result arrived above.

$v(t)$ and the plot of real part of its Fourier transform are shown in Fig. 14.4-9. The imaginary part of Fourier transform is zero.

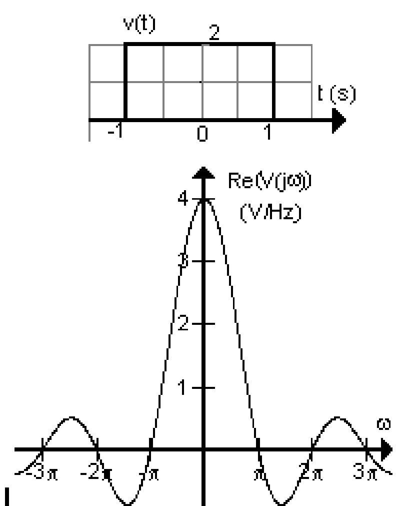


Fig. 14.4-9 Spectrum of Real Part of Fourier Transform in Example : 14.4-4

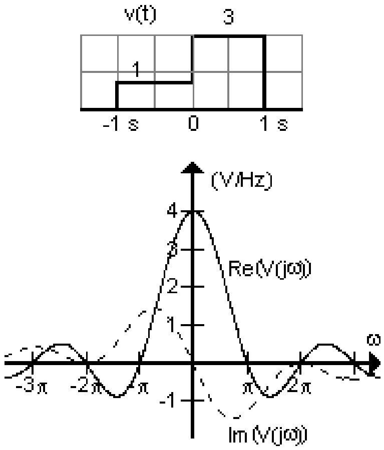


Fig. 14.4-10 Waveform and Spectra for Example : 14.4-5

Example : 14.4-5

Refer to Fig. 14.4-6. Let a new waveform be defined as $v(t) = 1.5\, v_1(t) + 0.5\, v_2(t)$ where $v_1(t)$ and $v_2(t)$ are as in Fig. 14.4-6. Find its Fourier transform and plot the real part and imaginary part of its Fourier transform against ω.

Solution

$$V(j\omega) = 1.5V_1(j\omega) + 0.5V_2(j\omega) = \left[2\frac{\sin(0.5\omega)}{(0.5\omega)}\right]\left[1.5e^{-j0.5\omega} + 0.5e^{j0.5\omega}\right]$$

$$= \left[2\frac{\sin(0.5\omega)}{(0.5\omega)}\right]\left[2\cos(0.5\omega) - j\sin(0.5\omega)\right]$$

The waveforms and spectra are shown in Fig. 14.4-10.

The series of examples from Example : 14.4-1 to Example : 14.4-5 has brought out certain interesting properties of Fourier transform of real function of time. The first three examples dealt with time-functions which had the property that $v(-t) = v(t)$. Their Fourier transforms had the property that they had only real parts. Moreover their real parts satisfied the condition that $V(-j\omega) = V(j\omega)$.

The fourth example dealt with a $v(t)$ which had the property that $v(-t) = -v(t)$ and its Fourier transform had only imaginary part. Moreover the imaginary part of Fourier transform satisfied the condition that $V(-j\omega) = -V(j\omega)$.

The fifth example dealt with a waveform that had no symmetry and its Fourier transform is found to have both real and imaginary parts. However its real part satisfies the condition that $\mathrm{Re}(V(-j\omega)) = \mathrm{Re}(V(j\omega))$ and its imaginary part satisfies the condition that $\mathrm{Im}(V(-j\omega)) = -\mathrm{Im}(V(j\omega))$.

We look into these symmetry properties of Fourier transform in the next section.

14.5 Symmetry Properties of Fourier Transforms

There is nothing wrong mathematically in $v(t)$ possessing an imaginary part. However, it will not be a *physical* waveform then. We deal with physical waveforms in Circuit Theory. Hence we deal with *real functions of time* exclusively. The Fourier transform of a real $v(t)$ has many interesting symmetry properties.

Conjugate Symmetry Property

Let $v(t)$ be a real function of t and $V(j\omega)$ its Fourier transform. Then $V(-j\omega) = V^*(j\omega)$. This can be shown as follows.

$$V(j\omega) = \int_{-\infty}^{\infty} v(t)\, e^{j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} v(t)\left[\cos\omega t + j\sin\omega t\right] dt$$

$$= \int_{-\infty}^{\infty} v(t)\cos\omega t\, dt + j\int_{-\infty}^{\infty} v(t)\sin\omega t\, dt$$

$$V(-j\omega) = \int_{-\infty}^{\infty} v(t)\cos(-\omega t)\, dt + j\int_{-\infty}^{\infty} v(t)\sin(-\omega t)\, dt$$

$$= \int_{-\infty}^{\infty} v(t)\cos\omega t\, dt - j\int_{-\infty}^{\infty} v(t)\sin\omega t\, dt$$

$$\therefore V(-j\omega) = V^*(j\omega)$$

This implies that $Re(V(-j\omega)) = Re(V(j\omega))$ and $Im(V(-j\omega)) = -Im(V(j\omega))$. It also implies that $|V(-j\omega)| = |V(j\omega)|$ and $\phi(-j\omega) = -\phi(j\omega)$ where $\phi(j\omega)$ is the angle of $V(j\omega)$.

Symmetry properties of Fourier transform of a real $v(t)$

Hence, *for a real function of time (i) the real part of its Fourier transform is an even function of frequency and imaginary part of its Fourier transform is an odd function of frequency and (ii) the magnitude of its Fourier transform is an even function of frequency while phase of its Fourier transform is an odd function of frequency*.

All the five examples in the previous section illustrate these properties.

Fourier Transform of an Even Time-Function

If $v(t)$ is an even function of t , $v(-t) = v(t)$. But by the time-reversal property of Fourier transforms we have that Fourier transform of $v(-t)$ is $V(-j\omega)$ which is $V^*(j\omega)$ by conjugate symmetry property. Therefore, for an *even* $v(t)$, $V(j\omega)$ must be equal to its own conjugate. That will be possible only if $V(j\omega)$ has a zero imaginary part. Therefore, *for a real and even* $v(t)$ *the Fourier transform* $V(j\omega)$ *is real and even on* ω. The first three examples in the Section 14.4 illustrate this aspect.

Symmetry properties of Fourier transform of a real even $v(t)$

Fourier Transform of an Odd Time-Function

If $v(t)$ is an odd function of t , $v(-t) = -v(t)$. But by the time-reversal property of Fourier transforms we have that Fourier transform of $v(-t)$ is $V(-j\omega)$ which is $V^*(j\omega)$ by conjugate symmetry property. Therefore, for an *odd* $v(t)$, $V(j\omega)$ must be equal to negative of its own conjugate. That will be possible only if $V(j\omega)$ has a zero real part. Therefore, *for a real and even* $v(t)$ *the Fourier transform* $V(j\omega)$ *is imaginary and odd on* ω. The fourth example in the Section 14.4 illustrates this aspect.

Symmetry properties of Fourier transform of a real odd $v(t)$

Fourier Transforms of Even Part and Odd Part of a Real Time-Function

If $v(t)$ is neither *even* nor *odd* , its Fourier transform will have both real and imaginary parts. Consider two auxiliary functions $v(t) + v(-t)$ and $v(t) - v(-t)$. The first is obviously *even* and the second is *odd*. Moreover $v(t)$ can be expressed in terms of these two as $0.5[v(t) + v(-t)] + 0.5[v(t) - v(-t)]$. Therefore, any real function $v(t)$ can be decomposed into $v_e(t)$ and $v_o(t)$ where $v_e(t)$ is its *even part* and $v_o(t)$ is its *odd part*. These are given by

$$v(t)=v_e(t)+v_o(t)$$

$$v_e(t)=\frac{v(t)+v(-t)}{2} \text{ and } v_o(t)=\frac{v(t)-v(-t)}{2}.$$

We apply linearity property of Fourier transforms to get

$$V(j\omega) = \mathrm{Re}(V(j\omega)) + j\,\mathrm{Im}(V(j\omega)) = V_e(j\omega)+V_o(j\omega)$$

But $v_e(t)$ is an *even* function of time and hence its Fourier transform must have only real part. Similarly $v_o(t)$ is an *odd* function of time and hence its Fourier transform must have only imaginary part. Therefore, we conclude that,

$$\textit{If } v(t) \Leftrightarrow V(j\omega), \textit{ then,}$$
$$Ev[v(t)] \Leftrightarrow \mathrm{Re}[V(j\omega)] \textit{ and } Od[v(t)] \Leftrightarrow \mathrm{Im}[V(j\omega)]$$
$$\textit{where } Ev[v(t)] \textit{ is the even part of } v(t) \textit{ and } Od[v(t)] \textit{ is its odd part.}$$

Fourier transforms of even and odd parts of a real $v(t)$

v(0) and *V*(j0)

The analysis integral evaluated at $t = 0$ reveals an interesting relation between the Fourier transform value at $\omega = 0$ and the total area-content of the time-function. They are equal. $V(j0)=\int_{-\infty}^{\infty} v(t)e^{j0t}dt = \int_{-\infty}^{\infty} v(t)\,dt = Total\ area-content\ of\ v(t)$.

Hence Fourier transform of a real time-function can not have an imaginary part at $\omega = 0$.

Similarly the synthesis integral evaluated at $t = 0$ reveals that the value of time-function at $t = 0$ is proportional to the total area-content of Fourier transform in frequency-domain. Duality property is at play here.

$$v(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)e^{j\omega 0}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)\,d\omega = \frac{Total\ area-content\ of\ V(j\omega)}{2\pi}$$

$V(j\omega)$ for a real $v(t)$ is conjugate symmetric. Hence the total area under $V(j\omega)$ will be the same as the total area under $Re(V(j\omega))$ and $v(0)$ = (Total area under real part of Fourier transform)/2π.

$V(j0)$ = Area under $v(t)$

$v(0)$ = (Area under real part of Fourier transform)/2π

14.6 Time-Scaling Property and Fourier Transform of Impulse Function

We are accustomed to the fact that when recorded music is played back at a higher speed, the treble content in the music increases and the bass content decreases. The operation that takes place in this case is compression of the signal in time-domain.

Consider a time-domain signal $v(t)$ and let $V(j\omega)$ be its Fourier transform. We define a new signal $v'(t)$ such that $v'(t) = v(at)$ where a is a real number. This means that at any t the value of $v'(t)$ is the value that $v(t)$ assumes at time instant at. If a is a positive real number greater than 1, the signal $v'(t)$ will be a compressed version of $v(t)$, and, if a is between 0 and 1 it will be an expanded version of $v(t)$. If a is a negative number, the signal $v'(t)$ will undergo a time reversal in addition to compression or expansion. These aspects are illustrated in Fig. 14.6-1. The original signal $v(t)$ in (a) of this figure is time-scaled by 2 in (b) and by 1/1.5 in (c). In (d) it is time-scaled by –2 and therefore undergoes compression as well as mirror reflection.

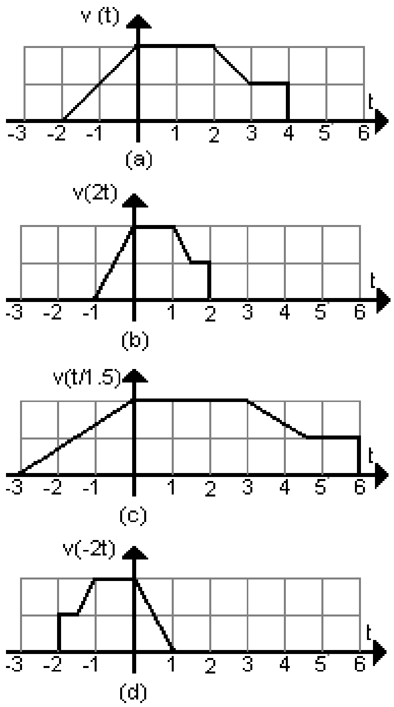


Fig. 14.6-1 Waveforms Illustrating Time-Scaling (a) Original Waveform (b) Time-Scaling by 2 (c) Time-Scaling by 1/1.5 (d) Time-Scaling by -2

Notice that when a waveform is scaled in time-domain its normalised energy changes by a factor $1/|a|$. Hence a compressed waveform will have lesser energy and expanded one will have higher energy.

What happens in frequency-domain when a signal $v(t)$ is time-scaled?

$$v(at) \Leftrightarrow \int_{-\infty}^{\infty} v(at)\, e^{j\omega t} dt$$

$$\text{Substitute } t' = at. \text{ Then, if } a \text{ is } +ve,$$

$$v(at) \Leftrightarrow \frac{1}{a}\int_{-\infty}^{\infty} v(t')\, e^{j\frac{\omega}{a}t'} dt'$$

$$\text{If } a \text{ is } -ve,\ t' \to \infty \text{ as } t \to -\infty \text{ and } t' \to -\infty \text{ as } t \to \infty$$

$$\therefore v(at) \Leftrightarrow -\frac{1}{a}\int_{-\infty}^{\infty} v(t')\, e^{j\frac{\omega}{a}t'} dt'$$

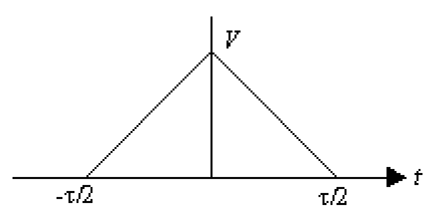
$$\therefore v(at) \Leftrightarrow \frac{1}{|a|} V(j\frac{\omega}{a})$$

Thus compression in time-domain ($a > 1$) is accompanied by expansion in frequency-domain, *i.e.*, the spectral amplitude density distribution shifts towards higher frequencies. Similarly expanding a waveform in time-domain results in concentration of spectral density towards lower frequencies. We consider an example to illustrate this.

Example : 14.6-1

A triangular pulse of amplitude V and duration τ sec is placed in the interval [$-\tau/2$, $\tau/2$] in the time-domain. (i) Derive an expression for its Fourier transform. (ii) Plot its spectrum for a special case of $V = 1$ and $\tau = 2$sec. (iii) Repeat (ii) if the waveform is time-scaled by a factor of 4.

Solution



This function is an even function of time. Its Fourier transform will have only real part. If $v(t)$ is even, $v(-t) = v(t)$

$$V(j\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t}dt = \int_{0}^{\infty}\left[v(t)e^{-j\omega t} + v(-t)e^{j\omega t}\right]dt$$

$$\therefore V(j\omega) = \int_{0}^{\infty} v(t)\left[e^{-j\omega t} + e^{j\omega t}\right]dt = 2\int_{0}^{\infty} v(t)\cos\omega t\, dt$$

The function $v(t)$ can be expressed as $2V(1-2t/\tau)$ for $0 \le t \le \tau/2$ and zero for $t > \tau/2$.

$$\therefore V(j\omega) = 2\int_0^{\tau/2} V(1-\tfrac{2t}{\tau})\cos\omega t\, dt = 2V\int_0^{\tau/2}\cos\omega t\, dt - \frac{4V}{\tau}\int_0^{\tau/2} t\cos\omega t\, dt$$

$$= 2V\frac{\sin \omega\tau/2}{\omega} - \frac{4V}{\tau}\left[t\frac{\sin\omega t}{\omega}\bigg|_0^{\tau/2} - \int_0^{\tau/2}\frac{\sin\omega t}{\omega}dt\right]$$

$$= 2V\frac{\sin \omega\tau/2}{\omega} - \frac{4V}{\tau}\left[t\frac{\sin\omega t}{\omega}\bigg|_0^{\tau/2} - \frac{\cos\omega t}{\omega^2}\bigg|_0^{\tau/2}\right] = \frac{4V}{\omega^2\tau}\left[1-\cos \omega\tau/2\right]$$

$$= \frac{V\tau}{2}\left(\frac{\sin \omega\tau/4}{\omega\tau/4}\right)^2 = (\text{Pulse Area})\times\left(\frac{\sin \omega\tau/4}{\omega\tau/4}\right)^2$$

The required plots are shown in Fig. 14.6-2.

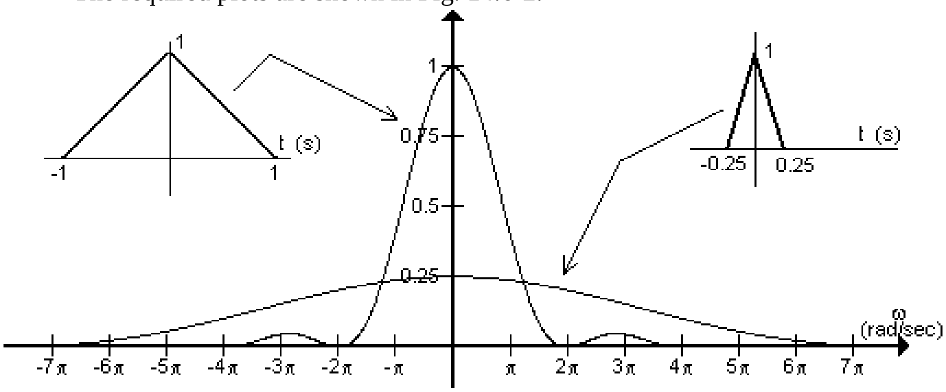


Fig. 14.6-2 Spectral Plots for Triangular Pulses in Example : 14.6-1

It is instructive to compare the spectrum of a rectangular pulse and a triangular pulse. Refer to the spectrum in Fig. 14.3-2 and compare it with the spectrum in Fig. 14.6-2.

We observe that both spectra are more concentrated in the low frequency range in the principal lobe of the spectral plot.However, the side lobe level is much less in the case of a triangular pulse than rectangular pulse. Side lobe levels indicate the extent to which high frequency components are needed to synthesize the waveform.

We note that a rectangular pulse needs considerable help from high frequency sinusoids for its synthesis than a triangular pulse needs. This is indeed expected since a rectangular pulse contains a jump discontinuity – the highest speed at which a waveform can change.

High speed changes in a waveform requires high frequency sinusoids for its construction.

The spectral plots clearly illustrate that compression in time-domain has resulted in lowering of spectral content at $\omega = 0$ (we remember that this value must be equal to the area-content of $v(t)$) and shifting of spectral content to higher frequency ranges. The spectrum attains its first zero at $4\pi/\tau$ rad/sec. When τ is decreased as a result of time-scaling, this value increases as seen in Fig. 14.6-2.

Compressing a Triangular Pulse in Time-Domain with its Area-Content Constant

The scaling factor of $1/|a|$ in $v(at) \Leftrightarrow 1/|a|\ \ V(j\omega/a)$ appears due to the fact that we did not change the area-content of the waveform while compressing it or expanding it. If we scale the amplitude by the same scaling factor that we use for time-scaling the waveform we get will be $av(at)$ and its Fourier transform will be $V(j\omega/a)$.

Let $v(t)$ be a triangular pulse of width 2τ and height $1/\tau$, located in the interval $[-\tau,\tau]$ in the time-domain. Its area-content is 1.Its Fourier transform $V(j\omega) = 1 \times [(\sin 0.5\omega\tau)/(0.5\omega\tau)]^2$. What happens to the Fourier transform as we let $\tau \to 0$ and pulse height $\to \infty$ while keeping its area at unity? Since (sin x)/x approaches 1 when $x \to 0$, $V(j\omega) \to 1$ as $\tau \to 0$.

But a time-domain waveform that has an infinitesimal width and infinite height with an area-content of unity has been defined as unit impulse function. Therefore, the Fourier transform of $\delta(t)$ is 1 for all ω. *i.e.,* $\delta(t) \Leftrightarrow 1$

Fourier transform of unit impulse function is unity.

The time-scaling property of Fourier transform in effect tells us that we can not *confine* a waveform to narrower regions in time-domain without allowing it to spread out in frequency-domain.*Time-domain waveforms are claustrophobic – they react by leaking out to higher frequency ranges in frequency-domain when they are confined to smaller intervals in time-domain.*

Waveforms and claustrophobia ?

Similarly, waveforms are claustrophobic in frequency-domain too – they react by spreading out in time-domain when they are confined in frequency-domain to narrower

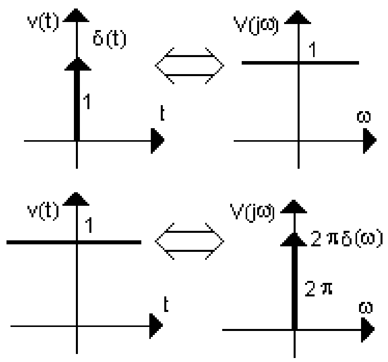


Fig. 14.6-3 Fourier Transforms of Unit Impulse and Unit Constant Functions

ranges of frequency. This must be clear if we consider time-scaling with $|a| < 1$. It follows from the Duality property of Fourier transforms too.

The narrowest confinement a waveform can be subjected to in time-domain is to confine it in an infinitesimal interval – this happens in the case of $\delta(t)$. *And, to synthesize it we need sinusoids of all frequencies from 0 to ∞ with equal strength for all sinusoids.* The narrowest confinement in time-domain results in widest expansion in frequency-domain.

Duality property of Fourier transform states that for every $v(t) \Leftrightarrow V(j\omega)$ pair there exists another Fourier transform pair $V(-jt) \Leftrightarrow 2\pi v(j\omega)$. We apply this property to the Fourier transform pair we have just now arrived at, *i.e.*, to $\delta(t) \Leftrightarrow 1$. We get $1 \Leftrightarrow 2\pi\,\delta(j\omega)$. This implies that, the Fourier transform of a *constant time-function* $v(t) = 1$ for all t, is an impulse of magnitude 2π located at $\omega = 0$ in the frequency-domain. Thus, widest expansion in time-domain is accompanied by narrowest confinement in frequency-domain.

These two Fourier transform pairs are shown in Fig. 14.6-3.

Unit impulse function and unit constant functions violate Dirichlet's conditions. Yet they have Fourier transforms. We worked out the Fourier transform of unit impulse function as a limiting case of a triangular pulse. However a straight application of analysis integral will also give us the same result. Uniqueness of Fourier transform pair assures us that the synthesis integral should give us $\delta(t)$. Therefore, applying the synthesis equation on Fourier transform of unit impulse function, we get,

$$\int_{-\infty}^{\infty} e^{j\omega t} d\omega = 2\pi\delta(t)\ .$$

And, by applying analysis equation on Fourier transform of unit constant function, we get,

$$\int_{-\infty}^{\infty} e^{-j\omega t} dt = 2\pi\delta(j\omega)\ .$$

Both the integrals are improper integrals and can not be evaluated otherwise.

14.7 Fourier Transforms of Periodic Waveforms

We arrived at the Fourier transform of $\delta(t)$ against the background of time-domain compression of waveforms in the last section. Further, by applying duality principle on this Fourier transform, we derived the Fourier transform for $v(t) = 1$. Then we tried to obtain the Fourier transform by applying analysis integral on $v(t) = 1$ and ended up with an improper integral. We remembered the theorem of uniqueness of Fourier transforms and assigned $2\pi\delta(j\omega)$ to this integral since we had seen by other means that this is the Fourier transform of $v(t) = 1$. This integral is

$\int_{-\infty}^{\infty} e^{-j\omega t} dt = 2\pi\delta(j\omega)$. We observe that the variable of integration is t and hence any substitution for ω in integrand will simply appear as argument of impulse on the right side. We choose to substitute $\omega = \omega - \omega_0$ where ω_0 is a specific value of ω. Then,

$$\int_{-\infty}^{\infty} e^{j(\omega-\omega_0)t}\, dt = 2\pi\delta(j[\omega-\omega_0])\ .$$

But the integral on the left side is the Fourier analysis integral for the time-function $e^{j\omega_0 t}$ since $\int_{-\infty}^{\infty} e^{j(\omega-\omega_0)t}\, dt = \int_{-\infty}^{\infty} \left[e^{j\omega_0 t}\right] e^{j(\omega-\omega_0)t}\, dt = FT\ of\ e^{j\omega_0 t}$.

Fourier transforms of $e^{j\omega_o t}$ and $e^{-j\omega_o t}$

Therefore,

$e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega-\omega_0)$ and by a similar argument $e^{-j\omega_0 t} \Leftrightarrow 2\pi\delta(j[\omega+\omega_0])$

We are only one step away from finding out the Fourier transforms of sin ωt and cos ωt.

$$\cos\omega_0 t = \frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2};\sin\omega_0 t = \frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}$$
$$\therefore \cos\omega_0 t \Leftrightarrow \pi\delta(j[\omega-\omega_0])+\pi\delta(j[\omega+\omega_0])$$
$$\sin\omega_0 t \Leftrightarrow -j\pi\delta(j[\omega-\omega_0])+j\pi\delta(j[\omega+\omega_0])$$

Fourier transforms of $\sin\omega_0 t$ and $\cos\omega_0 t$

These are periodic functions with unit amplitude starting at -∞ and proceeding to ∞ in the time-domain. They possess infinite energy. They are 'finite power' waveforms since they have a finite average power of 0.5 W over a cycle. They are not absolutely integrable and hence do not satisfy Dirichlet's conditions. Yet, they have Fourier transforms. Only that their Fourier transforms are pair of impulses located at $\pm\omega_0$ with each impulse magnitude at π volts (assuming they are voltage functions).

We remember that Fourier transform gives the density of spectral amplitude at a frequency. A cosine wave at ω_0 rad/sec has only one frequency component and all its amplitude, *i.e.*, 1 unit , is concentrated at ω_0. Hence the amplitude density at $\pm\omega_0$ (in two-sided exponential representation) must be ∞ and it must be zero at other frequency points. Density integrated must give amplitude. Therefore, though the density function is ∞ at $\pm\omega_0$ and zero elsewhere, its integral must yield 1 unit with 2π factor in inverse Fourier transform accounted. The impulse representation of Fourier transform of periodic functions is justified from this viewpoint too.

A non-sinusoidal periodic waveform can be represented as sum of harmonically related sine and cosine functions.

Hence a periodic waveform has a Fourier transform as well as a Fourier series.

Its Fourier transform will be a train of impulse functions located at harmonically related frequency points with the magnitude of impulses equal to magnitude of Fourier series coefficients.

The Fourier transforms of cosine and sine functions are shown in Fig. 14.7-1.

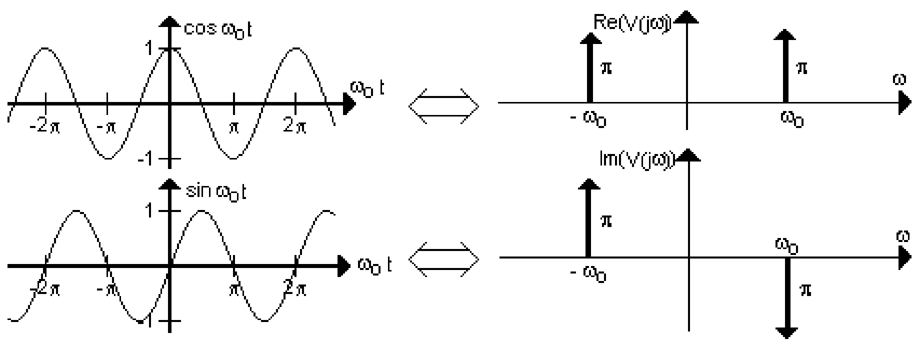


Fig. 14.7-1 Fourier Transforms of Unit Cosine and Unit Sine Waveforms

14.8 Fourier Transforms of Some Semi-Infinite Duration Waveforms

Inputs, even if they are periodic waveforms, are switched on to the circuit at some finite time instant. This fact is usually brought out by specifying an input function as $f(t)$ $u(t)$ where $f(t)$ is a periodic or aperiodic waveform. A dc voltage of V applied to the circuit from t = 0 is specified as $Vu(t)$ and a unit amplitude cosine wave applied to the circuit from t = 0 is specified as cos (ωt) $u(t)$.

We developed the Fourier transforms for periodic waveforms, including the one with infinite period, *i.e.*, a constant function, in the last section. We study the effect of switching at t = 0 on their Fourier transforms in this section.

Fourier transform of $e^{-\alpha t}u(t)$

Let $v_1(t) = e^{-\alpha t}u(t)$. We find its Fourier transform by straightforward application of the analysis integral.

Fourier transform of $e^{-\alpha t}u(t)$

$$V_1(j\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t}dt = \int_0^{\infty} e^{-\alpha t}e^{-j\omega t}dt = \frac{1}{j\omega+\alpha}$$

$$\text{Real Part } = \frac{\alpha}{\alpha^2+\omega^2} \text{ and Imaginary part} = \frac{-\omega}{\alpha^2+\omega^2} \qquad (14.8\text{-}1)$$

$$\text{Magnitude } = \frac{1}{\sqrt{\alpha^2+\omega^2}} \text{ and Phase } = -\tan^{-1}\left(\frac{\omega}{\alpha}\right)$$

Note that (i) real part of Fourier transform is *even* (ii) imaginary part of Fourier transform is *odd* (iii) the magnitude of Fourier transform is *even* (iv) the phase of Fourier transform is *odd*.

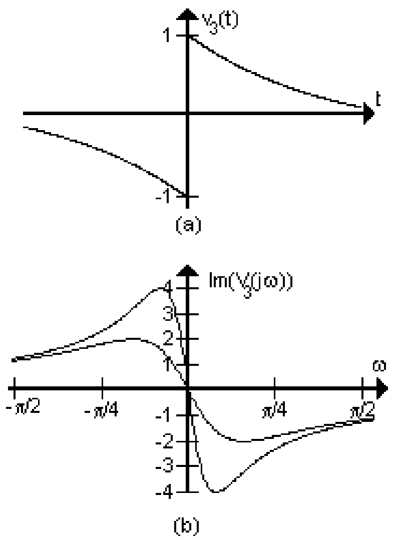


Fig. 14.8-1 (a) Waveform of $[e^{-\alpha t}u(t) - e^{\alpha t}u(-t)]$ and (b) its Fourier Transform for α=0.5 and 0.25

Now let us consider a function $v_2(t)$ defined as $v_2(t) = e^{\alpha t}u(-t)$. It is easily seen that $v_2(t)$ is $v(-t)$, *i.e.*, a mirror reflected version of $v(t)$. Using the time-reversal property and conjugate symmetry property of Fourier transforms we write its Fourier transform as

$$V_2(j\omega) = = \frac{1}{-j\omega + \alpha}$$

$$\text{Real Part } = \frac{\alpha}{\alpha^2+\omega^2} \text{ and Imaginary part} = \frac{\omega}{\alpha^2+\omega^2} \qquad (14.8\text{-}2)$$

$$\text{Magnitude } = \frac{1}{\sqrt{\alpha^2+\omega^2}} \text{ and Phase } = \tan^{-1}\left(\frac{\omega}{\alpha}\right)$$

We consider a third function $v_3(t)$ defined as $v_3(t) = v_1(t) - v_2(t) = e^{-\alpha t}u(t) - e^{\alpha t}u(-t)$. $V_3(j\omega)$ is obtained by subtracting $V_2(j\omega)$ from $V_1(j\omega)$. Hence $V_3(j\omega)$ will have only imaginary part. We note that $v_3(t)$ is indeed *odd*.

$$V_3(j\omega) = V_1(j\omega) - V_2(j\omega)$$
$$= -j\frac{2\omega}{\alpha^2+\omega^2} \qquad (14.8\text{-}3)$$

The waveform $v_3(t)$ and the imaginary part of its Fourier transform for two values of α = 0.5 and 0.25 are shown in Fig. 14.8-1. Note that in both cases the imaginary part of Fourier transform goes through origin – in fact this will be true for any non-zero value of α as Eqn. 14.8-3 would indicate.

Fourier Transform of Signum Function

Signum function defined

The function $[e^{-\alpha t}u(t) - e^{\alpha t}u(-t)]$ takes on an interesting shape as $\alpha \to 0$. It becomes +1 for $t > 0$ and –1 for $t < 0$. This is defined as the *Signum function of t* and is indicated by sgn (t).

$$\text{sgn}(t) = \begin{cases} -1 \text{ for } -\infty < t < 0 \\ \textit{undefined} \text{ for } t = 0 \\ 1 \quad \text{for } 0 < t < \infty \end{cases} \qquad (14.8\text{-}4)$$

sgn(t) does not satisfy Dirichlet's conditions. But $v(t) = [e^{-\alpha t}u(t) - e^{\alpha t}u(-t)]$ does for +ve values for α. Therefore we express sgn(t) as the limit of $v(t)$ defined as above as $\alpha \to 0$. It is only a *limit* and hence α never becomes zero; it only approaches zero. Therefore its Fourier transform given by Eqn. 14.8-3 always goes through origin in frequency-domain. We now express the Fourier transform of sgn(t) as the limit of Fourier transform of $v(t)$ as $\alpha \to 0$.

$$\therefore \text{sgn}(t) \Leftrightarrow \lim_{\alpha\to 0}\left[-j\frac{2\omega}{\alpha^2+\omega^2}\right] = \frac{2}{j\omega} \qquad (14.8\text{-}5)$$

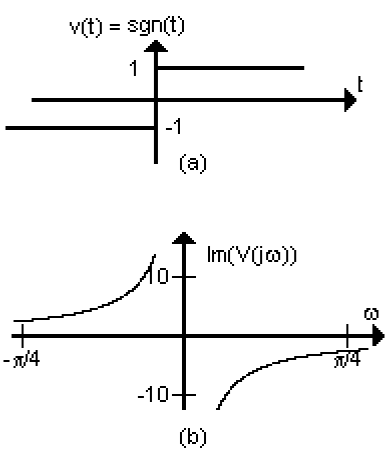


Fig. 14.8-2 Signum Function and its Fourier Transform

This equation suggests that Fourier transform of sgn(t) goes to ∞ at ω = 0. It doesn't. It doesn't because Fourier transform of $[e^{-\alpha t}u(t) - e^{\alpha t}u(-t)]$ is a continuous function which goes through origin for all positive values of α and when we take the limit we only make α arbitrarily close to 0 ; but never 0. Therefore Fourier transform of sgn(t) is the limit in Eqn. 14.8-5; but its second form *i.e.*, ($2/j\omega$) can be used only for $\omega \neq 0$. This is also supported by the fact that the value $V(j0)$ of a Fourier transform is equal to the *total area-content* of $v(t)$. The total area-content of sgn(t) is obviously zero. Therefore Fourier transform of sgn(t) should have zero value at ω = 0. Signum function and its Fourier transform are shown in Fig. 14.8-2.

Fourier Transform of Unit Step Function

The unit step function $v(t) = u(t)$ is a semi-infinite duration, infinite energy waveform that violates Dirichlet's conditions. It has a Fourier transform if it is interpreted as a limiting case of another time-function.

Consider $v'(t)$ defined as $v'(t) = 0.5 + 0.5[e^{-\alpha t} u(t) - e^{\alpha t} u(-t)]$. When we send $\alpha \to 0$ the second term in this function approaches 0.5 sgn(t) and $v'(t)$ approaches $u(t)$. Therefore, the Fourier transform of $u(t)$ is the limit of Fourier transform of $v'(t)$ as $\alpha \to 0$. Therefore, we get ,

$$u(t) \Leftrightarrow \pi\delta(j\omega)+0.5\lim_{\alpha\to 0}\left[\frac{-j2\omega}{\alpha^2+\omega^2}\right]=\pi\delta(j\omega)+\frac{1}{j\omega} \qquad (14.8\text{-}6)$$

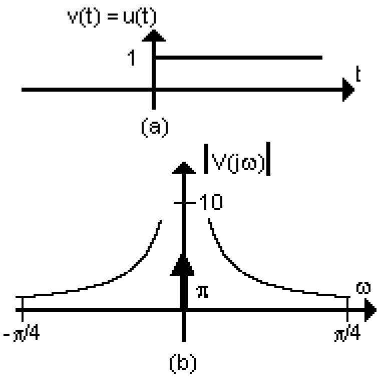


Fig. 14.8-3 Unit Step Function and Magnitude of its Fourier Transform

We have to remember that α never becomes equal to zero and hence the $1/j\omega$ component should never be evaluated at $\omega = 0$. Instead, the value of second term at $\omega = 0$ should be correctly evaluated as zero.

The first component in the Eqn. 14.8-6 indicates that there is a constant component of 0.5 in $u(t)$ – it is indeed so since the average value of $u(t)$ is 0.5. The $u(t)$ function and its Fourier transform magnitude are shown in Fig. 14.8-3.

Fourier Transform of Functions of the Form $e^{(-\alpha+j\omega_o)t}\times v(t)$

Let $v'(t)=e^{(-\alpha+j\omega_0)t}\times v(t)$. α is a positive real number. The Fourier transform of $v'(t)$ is

$$\begin{aligned}V'(j\omega)&=\int_{-\infty}^{\infty} v(t)e^{(-\alpha+j\omega_0)t}e^{-j\omega t}dt\\&=\int_{-\infty}^{\infty} v(t)e^{-[\alpha+j(\omega-\omega_0)]t}dt \qquad (14.8\text{-}7)\\&=V[\alpha+j(\omega-\omega_0)]\end{aligned}$$

Fourier transform of $e^{(-\alpha+j\omega)t}v(t)$

This implies that the required Fourier transform can be obtained by replacing every ($j\omega$) in the Fourier transform of $v(t)$ by $\alpha + j(\omega\text{-}\omega_0)$. We identify two special cases here. In the first case we make $\omega_0 = 0$. Then

$$e^{-\alpha t}v(t) \Leftrightarrow V(\alpha+j\omega) \qquad (14.8\text{-}8)$$

Fourier transform of $e^{-\alpha t}v(t)$

We had derived the Fourier transform of $e^{-\alpha t} u(t)$ in an earlier sub-section and the result was $1/(\alpha + j\omega)$. But it should be $\pi\delta(\alpha + j\omega) + 1/(\alpha + j\omega)$ according to Eqn. 14.8-8 because Fourier transform of $u(t)$ is $\pi\delta(\omega) + 1/j\omega$. How do we explain the difference? The function $\pi\delta(\alpha + j\omega)$ is 0 for $(\alpha + j\omega) < 0$ and $(\alpha + j\omega) > 0$. It has an area-content of π located at $(\alpha + j\omega) = 0$. But no value of ω in the range -∞ to +∞ can make $(\alpha + j\omega)$ equal to zero because α is a *real number*. Therefore, this impulse component will never come into play and hence need not be carried in the Fourier transform function. Therefore, Fourier transform of $e^{-\alpha t} u(t)$ is $1/(\alpha + j\omega)$ by Eqn. 14.8-8 too. $u(t)$ is an infinite-energy waveform ; $e^{-\alpha t} u(t)$ with positive real α is a finite-energy waveform. This explains the presence of impulse content in Fourier transform of $u(t)$ and its absence in the Fourier transform of $e^{-\alpha t} u(t)$.

The second case is the one with $\alpha = 0$. Then, $e^{j\omega_0 t}v(t) \Leftrightarrow V[j(\omega-\omega_0)]$. This means that the Fourier transform gets *shifted in frequency-domain* when the time-domain function is multiplied by a complex exponential function. This is termed as the *Frequency-shifting Property* of Fourier transforms.

Frequency shifting property of Fourier transform

Now we multiply $u(t)$ by $0.5(e^{j\omega_0 t}+e^{-j\omega_0 t})$ to get $\cos\omega_0 t\ \ u(t)$. The Fourier transform of this function is

$$\cos\omega_0 t\times u(t)\Leftrightarrow\frac{\pi}{2}[\delta(j[\omega-\omega_0])+\delta(j[\omega+\omega_0])]+\frac{j\omega}{(j\omega)^2+\omega_0^2} \qquad (14.8\text{-}9)$$

Fourier transform of $[\cos\omega_o t]u(t)$

We have used the Fourier transform of $u(t)$ in deriving the above Fourier transform and hence the second term on the right side should not be evaluated at ω_0. This Fourier transform indicates that there is a pair of impulses located at $\pm\omega_0$. This implies that there is a periodic component 0.5 $\cos\omega_0 t$ in this signal. It is indeed so since we can write $\cos\omega_0 t\times u(t)$ as $[0.5+0.5\text{sgn}(t)]\cos\omega_0 t$.

Fourier transform of $\sin\omega_0 t\ u(t)$ can be similarly derived as

Fourier transform of $[\sin\omega_o t]u(t)$

$$\sin\omega_0 t\times u(t)\Leftrightarrow j\frac{\pi}{2}[-\delta(j[\omega-\omega_0])+\delta(j[\omega+\omega_0])]+\frac{\omega_0}{(j\omega)^2+\omega_0^2} \quad (14.8\text{-}10)$$

Fourier transform of $e^{-\alpha t}\cos\omega_0 t\ u(t)$ is obtained by using Eqn. 14.8-7 along with Eqn. 14.8-9. The resulting Fourier transform is

Fourier transform of the function $[e^{-\alpha t}\cos\omega_o t]u(t)$.

$$e^{-\alpha t}\cos\omega_0 t\times u(t)\Leftrightarrow\frac{(\alpha+j\omega)}{(\alpha+j\omega)^2+\omega_0^2} \quad (14.8\text{-}11)$$

Similarly the Fourier transform of exponentially damped sine wave can be obtained as

Fourier transform of the function $[e^{-\alpha t}\sin\omega_o t]u(t)$.

$$e^{-\alpha t}\sin\omega_0 t\times u(t)\Leftrightarrow\frac{\omega_0}{(\alpha+j\omega)^2+\omega_0^2} \quad (14.8\text{-}12)$$

In both cases we have dropped the impulse components in the spectrum since those impulses can not come into action for any legitimate value of ω. Notice that damped sinusoids are aperiodic waveforms with finite energy whereas undamped sinusoids are aperiodic waveforms with infinite energy.

14.9 Zero-State Response by Frequency-Domain Analysis

Two questions were raised in the beginning of this chapter.

(i) Can an aperiodic waveform be resolved into a sum of sinusoids? If yes, which are the frequencies that are present in the signal decomposition?

(ii) Can steady-state frequency response function help us to get the output when such an aperiodic waveform is applied to the circuit? If yes, how do we explain a *steady-state* function yielding a *transient response* term?

The first question was answered in the previous sections. The second question is addressed in Section 14.9 and in the subsequent sections.

We have seen that all practical waveforms, periodic or aperiodic, possess frequency-domain descriptions in the form of Fourier transform. All waveforms can be synthesized from sinusoids of different frequencies by adding them up. The sinusoids that constitute a periodic waveform as well as an aperiodic waveform are *periodic* and start from -∞ and go to +∞ in the time-domain. *All the details in an aperiodic waveform are constructed by such everlasting sinusoidal signals interfering with each other constructively and destructively in various time-intervals. Thus, an infinite number of periodic sinusoids synthesize the aperiodic nature of an aperiodic waveform.*

Aperiodic nature of a waveform gives rise to transient response in a linear circuit when it is applied to the circuit. We are considering a linear circuit and such a circuit obeys superposition principle as far as *zero-state response* is concerned. Hence, the zero-state response to the aperiodic input can be obtained by summing up the zero-state responses that the circuit will display for each component in a set of waveform components that constitute the aperiodic waveform. The set of waveform components we have in mind are those infinite number of sinusoids with frequencies from 0 to ∞ with infinitesimal amplitudes (for finite energy signals) which go into the synthesis of the aperiodic waveform. But, each such component is *periodic and everlasting* and, hence, will produce only a steady-state response. Therefore, we conclude that the *zero-state response* (including the transient response components) in a linear circuit when driven by an aperiodic waveform can be obtained as a superposition of many (possibly infinite) *sinusoidal steady-state response components*.

Let $v_i(t)$ be the input signal to a *linear* circuit and let $v_o(t)$ be its output. We may assume them to be voltage variables for being specific. Let $V_i(j\omega)$ and $V_o(j\omega)$ be the corresponding Fourier transforms. Consider an infinitesimal band of frequencies $\Delta\omega$ around a frequency ω. The sum of amplitudes of all complex exponential components contributing to $v_i(t)$ from this band is $V_i(j\omega)\times\Delta\omega/2\pi$. For a sufficiently small $\Delta\omega$ all those components may be taken to have same frequency ω. Hence the complex exponential input due to components from this band is $(V_i(j\omega)\times\Delta\omega/2\pi)e^{j\omega t}$. The output due to this component is given by multiplying this input component by the value of *frequency response function* of the circuit. We had represented this function by $H(j\omega)$ earlier. Its magnitude gives the ratio of output amplitude to input amplitude. Its phase angle gives the phase by which the output leads the input.