## Introduction

A linear time-invariant circuit is completely characterized in time-domain by its impulse response $h(t)$. The sinusoidal steady-state frequency response function $H(j\omega)$ too can be obtained from impulse response with the help of convolution integral. $H(j\omega)$ was seen to contain $h(t)$ in a hidden form in the form of an equation $H(j\omega)=\int_{-\infty}^{\infty} h(t)\,e^{-j\omega t}\,dt$ in Chapter 12. That observation led us to the present part of the book that had the stated aim of showing that $H(j\omega)$ is an equally complete characterization of a linear time-invariant circuit in frequency-domain.

Chapter 13 and Chapter 14 have more or less settled the issue in the case of stable circuits. Chapter 13 showed that any arbitrary periodic input waveform with certain minimal constraints on them could be expanded in terms of harmonically related sinusoidal waveforms. The chapter further showed how the forced response part (and hence steady-state response part) of the output of a stable circuit can be constructed using this expansion and the $H(j\omega)$ of the circuit. Chapter 14 showed that any arbitrary transient waveform, which satisfies the Dirichlet's criteria, can be expanded in terms of sinusoidal waveforms of all frequencies from 0 to ∞ (Fourier Transforms). Further, it showed that zero-state response of a stable circuit to any such input can be constructed from its sinusoidal expansion and the $H(j\omega)$ function. Special emphasis was also placed on the point that even a waveform that does not satisfy Dirichlet's criteria can have a Fourier Transform since Dirichlet's criteria are only sufficient conditions and not necessary conditions. The standard unit step function $u(t)$ is an example.

Thus the Fourier transform technique has shown that the sinusoidal steady-state frequency response function $H(j\omega)$ is a complete characterization of a linear time-invariant stable circuit for all input functions that possess a Fourier transform.

However, Fourier transformation technique will not help us in dealing with inputs that are not absolutely integrable since the Fourier transform of such waveforms generally do not converge, and hence, may not exist. Moreover, even if Fourier transform exists for such waveforms, considerable ingenuity will be needed to find out the transform. Observe that we had to deal with $u(t)$ in a roundabout manner in order to find its Fourier transform in Chapter 14. Thus, the class of functions that are not absolutely integrable, *i.e.*, $\int_{-\infty}^{\infty} |v(t)|\,dt$ does not converge, do pose a problem to frequency-domain analysis of circuits using Fourier transform. What is evidently in need is a more powerful and more general version of signal expansion and a more general version of the system function $H(j\omega)$ so that we can claim the frequency-domain description of linear time-invariant circuits to be as complete as the time-domain description. This chapter deals with such a signal expansion technique – it is called the Laplace Transform.

**The sinusoidal steady-state frequency response function *H(jω)* which can be obtained from impulse response with the help of convolution integral is a complete characterization of a linear time-invariant stable circuit in frequency-domain as far as input functions which possess a Fourier transform are concerned.**

**However there are inputs for which Fourier transforms may not exist or are difficult to find. Further, we need a frequency-domain description for unstable circuits too. Thus Fourier description of linear time-invariant circuits is not a complete description.**

**Laplace transform completes the frequency-domain description of linear time-invariant circuits.**

Sinusoidal waveforms were the basis functions for expanding a signal in Fourier series and Fourier transforms. The reasons for choosing sinusoidal waveforms for this purpose were elucidated in the introductory portion of Chapter 13. We noted there that, the choice of basis function for expanding an input signal depends on the requirement that the forced response of a linear time-invariant circuit to the chosen function must be easy to determine. We would appreciate it if we could simplify multiply the function by a number (possibly complex) to obtain the forced response. But, then, complex exponential functions are eigen functions of linear time-invariant circuits, and, the forced response part of output is decided precisely by a multiplication with a complex number in the case of an eigen function input. Therefore, the complex exponential function $e^{st}$ is the basis function that we would like to use in expanding an arbitrary input waveform. Based on our decision to try simpler choices first, we used $s = j\omega$ and thereby limited our choice to pure sinusoidal functions drawn from the general class of complex exponential functions in Chapter 13 and Chapter 14. The result was the Fourier transform description of a time-domain waveform. But we meet with certain useful

Sinusoidal waveforms are the *basis functions* in Fourier description of circuits. Generalised complex exponential waveforms are the *basis functions* in Laplace transform description of circuits.

waveforms that refuse to yield to Fourier transformation. Hence we let the basis function be any general complex exponential function $e^{st}$ and try to obtain frequency-domain description ( *i.e.*, signal expansion) for much broader class of waveforms now.

The time-domain description of linear time-invariant circuit in the form of convolution integral does not shy away from unstable circuits. Convolution integral and other time-domain techniques apply to unstable and marginally stable circuits too. However, the impulse response of an unstable circuit contains natural response terms that grow with time and hence will not be absolutely integrable. Therefore, impulse response of an unstable circuit may not have a Fourier transform. Therefore, Fourier transform technique can not handle unstable circuits. Laplace transform technique can.

## 15.1 Circuit Response to Complex Exponential Input

Let the $n^{th}$ order differential equation describing a $n^{th}$ order linear time-invariant circuit be

$$\frac{d^n y}{dt^n}+a_{n-1}\frac{d^{n-1}y}{dt^{n-1}}+\cdots+a_1\frac{dy}{dt}+a_0 y=b_m\frac{d^m x}{dt^m}+b_{m-1}\frac{d^{m-1}x}{dt^{m-1}}+\cdots+b_1\frac{dx}{dt}+b_0 x \quad (15.1\text{-}1)$$

$y(t)$ is some circuit variable identified as the output variable and $x(t)$ is some independent voltage/current source function. Let $x(t) = 1\ e^{st}$ be a complex exponential function of unit amplitude and complex frequency $s = \sigma + j\omega$. Let $y(t) = \mathbf{A}\ e^{st}$ be the trial solution where $\mathbf{A}$ is a complex number to be determined. Substituting the trial solution in Eqn. 15.1-1, we get,

$$\left[s^n + a_{n-1}s^{n-1} + \cdots\cdots + a_1 s + a_0\right]\mathbf{A}e^{st} = \left[b_m s^m + b_{m-1}s^{m-1} + \cdots\cdots + b_1 s + b_0\right]e^{st}$$

$$\therefore \mathbf{A} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots\cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots\cdots + a_1 s + a_0}$$

Thus, when input to a linear time-invariant circuit is a complex exponential function $e^{st}$ , the output is the same complex exponential function multiplied by a complex number. Therefore, the complex frequency of output *remains same as that of input*. The output will have a different phase compared to that of input since $\mathbf{A}$ is a complex number in general and has an angle. The value of this complex scaling factor depends on the coefficients of circuit differential equation ( *i.e.*, on the circuit parameters) and the complex frequency $s$ of the input. In consonance with the symbol $H(j\omega)$ used for a similar complex number that relates the output to an input of $e^{j\omega t}$, we use the symbol $H(s)$ to represent this number $\mathbf{A}$ from this point onwards. Therefore,

when $x(t) = e^{st}$ in a linear time-invariant circuit, $y(t) = H(s)\ e^{st}$ , where

$H(s)$ - the generalised frequency response function of a linear time-invariant circuit

$$H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots\cdots + b_1 s + b_o}{s^n + a_{n-1}s^{n-1} + \cdots\cdots + a_1 s + a_o}\ .$$

But, which component of response is this? Since $x(t) = 1\ e^{st}$, the complex exponential function was taken to be applied to the circuit from $t = -\infty$ onwards. Therefore, there is only one component in response and that is the forced response. Therefore, the response given above is the forced response as well as the total response. But if $x(t) = 1\ e^{st}\ u(t)$ , then, the above expression yields the forced response component only. The natural response terms in zero-state response and the natural response terms in zero-input response have to be found out from initial conditions. However, those terms are also expected to be complex exponential functions since natural response terms of a linear time-invariant circuit are complex exponential functions.

The complex function $H(s)$ of a complex variable $s$ can also be written in polar form as $|\ H(s)|\ \angle\theta$ and in exponential form as $|\ H(s)|\ e^{j\theta}$ where $\theta$ is its angle. Therefore, the output $y(t)$ can be expressed as $y(t) = |\ H(s)|\ e^{st\text{-}j\theta} = |\ H(s)|\ e^{\sigma t}\ e^{j(\omega t+\theta)}$. $H(s)$ may be viewed as a *generalised frequency response function. Its magnitude gives the ratio between the amplitude of output complex exponential function and input complex exponential function. Its angle gives the phase angle by which the output complex exponential function leads the input complex exponential function.*

### Example : 15.1-1

A gated input function $v(t) = 2e^{0.2t}\cos 2t\, u(t)$ volts is applied across a series $RC$ circuit with $RC = 2$ seconds. The voltage across the capacitor is taken as the output. Determine the zero-state response of the output voltage.

**Solution**

The differential equation governing the voltage $v_o(t)$ across the capacitor in a series $RC$ circuit is $\frac{dv_o}{dt} + \frac{1}{RC}v_o = \frac{1}{RC}v$ where $v(t)$ is the input voltage. Therefore $H(s)$ $= \frac{1/RC}{s + 1/RC} = \frac{1}{1+sRC} = \frac{1}{1+2s}$.

$$v(t) = 2e^{0.2t}\cos 2t = 2e^{0.2t}\left[\frac{e^{j2t}+e^{-j2t}}{2}\right] = e^{(0.2+j2)t} + e^{(0.2-j2)t}$$

We apply superposition principle to determine the forced response to these two input components.

Forced response to $e^{(0.2+j2)t} = \left.\frac{1}{1+2s}\right|_{s=0.2+j2} \times e^{(0.2+j2)t} = \frac{1}{1.4+j4}\times e^{(0.2+j2)t}$

Forced response to $e^{(0.2-j2)t} = \left.\frac{1}{1+2s}\right|_{s=0.2-j2} \times e^{(0.2-j2)t} = \frac{1}{1.4-j4}\times e^{(0.2-j2)t}$

$$\begin{aligned}\therefore \text{Forced response to } 2e^{0.2t}\cos 2t &= \frac{1}{1.4+j4}\times e^{(0.2+j2)t} + \frac{1}{1.4-j4}\times e^{(0.2-j2)t}\\ &= 0.236[e^{(0.2+j2)t-j1.234} + e^{(0.2-j2)t+j1.234}]\\ &= 0.236\times e^{0.2t}\times[e^{j(2t-1.234)} + e^{-j(2t-1.234)}]\\ &= 0.236\times e^{0.2t}\times 2\cos(2t-1.234)\\ &= 0.472e^{0.2t}\cos(2t-70.71^0)\text{ volts}\end{aligned}$$

We need to find the zero-state response. The initial capacitor voltage is zero since we are trying to solve for zero-state response. Therefore $v_o(t) = Ae^{-0.5t} + 0.472\, e^{0.2t}\cos(2t-70.71^0)$ volts with $v_o(0^+) = 0$. Therefore $A = -0.472\cos(-70.71^0) = -0.156$.

Therefore, the zero-state response to $2e^{0.2t}\cos 2t\, u(t) = -0.156\, e^{-0.5t} + 0.472\, e^{0.2t}\cos(2t-70.71^0)$ volts. (See the side-note)

We had to add a natural response term $Ae^{-0.5t}$ and adjust the value of $A$ to meet the initial condition in the Example: 15.1-1 since the applied voltage was $2e^{0.2t}\cos 2t\,u(t)$ and not $2e^{0.2t}\cos 2t$. We could have avoided this step and obtained the zero-state response in one step if we could express the function $2e^{0.2t}\cos 2t\,u(t)$ as a sum of complex exponential functions of type $e^{j\omega t}$ – that would be a Fourier transform.

Fourier transform expresses a transient function as the sum of infinitely many sinusoidal waveforms that start at $-\infty$ and go up to $\infty$ in time-axis. But this waveform has no Fourier transform. It is a growing function of time.

But can it be expressed as a sum of complex exponential functions of type $e^{st} = e^{(\sigma+j\omega)t}$ with some non-zero value for $\sigma$? If it can be expressed that way, we can obtain the zero-state response to $2e^{0.2t}\cos 2t\,u(t)$ as the sum of forced response components to many complex exponential functions with the help of $H(s)$.

## 15.2 Expansion of a Signal in terms of Complex Exponential Functions

The signal $2e^{0.2t}\cos 2t\, u(t)$ does not have a Fourier transform because it is not absolutely integrable. We formulate the signal decomposition problem in a general form now. We define the kind of signals that we are going to use first.

Let $v(t) = f(t)\, u(t)$ be a right-sided function of time. We consider only right-sided functions of time in the analysis of linear time-invariant circuits since that is the kind of functions we apply to them and that is the kind of functions we get from them. The *underlying function* $f(t)$ is defined for all $t$ and may be non-zero for $t < 0$. For instance, if $f(t) = 1$ for all $t$ then $v(t)$ defined as $f(t)\, u(t)$ is a unit step function and stands for switching a 1 V dc source on to the circuit at $t = 0$.

If $v(t)$ is absolutely integrable - *i.e.*, if $\int_0^\infty f(t)dt$ is finite – and if $v(t)$ satisfies other Dirichlet's conditions, its Fourier transform will exist. If $v(t)$ is *not* absolutely integrable, its Fourier transform *may not* exist. In any case, we assume that the absolute value of the right-sided signal is *bounded by real exponential function* - *i.e.*, $|v(t)| < Me^{\alpha t}$ for all $t \geq 0$ with some value of $M$ and $\alpha$. For instance, the signal $v(t) = 1u(t)$ is bounded by $Me^{\alpha t}$ with any $M > 1$ and any $\alpha > 0$. The signal $v(t) = e^{\beta t}\, u(t)$ is bounded by $Me^{\alpha t}$ with any $M > 1$ and any $\alpha > \beta$ . Similarly, the signal $v(t) = e^{\beta t}$ $\cos(\omega t+\theta)\,u(t)$ is bounded by $Me^{\alpha t}$ with any $M > 1$ and any $\alpha > \beta$ . There are signals that can not be bounded in this sense. We do not deal with them in circuit analysis. We assume that the signals we deal with in this chapter are bounded in this sense.

Now, we define a new signal $v_1(t)$ by multiplying $v(t)$ with a scaling function $e^{-\sigma t}$ with the value of $\sigma$ equal to $\alpha$ or greater than $\alpha$ where $\alpha$ appears in the index of the *bounding exponential* $Me^{\alpha t}$.

$\therefore v_1(t) = v(t)e^{-\sigma t} = f(t)e^{-\sigma t}u(t)$ where $|v(t)| < Me^{\alpha t}$ and $\sigma \ge \alpha$ . This results in $v_1(t)$ becoming an absolutely integrable function. In fact $v_1(t)$ goes to zero as $t\to\infty$ since the factor $e^{-\sigma t}$ is chosen to overpower the maximum growth rate the function $v(t)$ may possibly exhibit and to make it a decaying function thereby. Therefore Fourier transform exists for $v_1(t)$.

$\therefore V_1(j\omega) = \int_{0^-}^{\infty} v(t)e^{-\sigma t}e^{-j\omega t}dt$ where the lower limit of integration is set at $0^-$ to handle impulse functions.

$$V_1(j\omega) = \int_{0^-}^{\infty} v(t)e^{-\sigma t}e^{-j\omega t}dt$$

$$= \int_{0^-}^{\infty} v(t)\,e^{-(\sigma+j\omega)t}dt$$

$$i.e., V_1(j\omega) = \int_{0^-}^{\infty} v(t)\,e^{-st}dt \quad \text{where } s = \sigma + j\omega = \text{the general complex frequency}$$

Laplace transform makes its appearance

Now this $V_1(j\omega)$ which is the *Fourier transform* of an exponentially scaled version of $v(t)$ with a particular value of $\sigma$ that appears in the index of the exponential scaling function is *defined* as the *Laplace transform* $V(s)$ *of* $v(t)$.

Fourier transform of $v_1(t)$ exists and the Fourier inverse integral converges to $v_1(t)$ for all $t$ only if $\sigma > \alpha$ . Therefore, $V_1(j\omega)$ exists only for $\sigma > \alpha$ . And since $V(s)$ is only another name for $V_1(j\omega)$, we conclude that,

If $v(t) = f(t)\,u(t)$ and $|v(t)| < Me^{\alpha t}$ for some $M$ and $\alpha$, then $V(s) = \int_{0^-}^{\infty} v(t)\,e^{-st}dt$ is its Laplace Transform where $s = \sigma + j\omega$ is the general complex frequency with $\sigma \ge \alpha$. The Laplace Transform exists and the inverse integral converges to $v(t)$ only for those values of $s$ that have $\mathrm{Re}(s) > \alpha$ . The region formed by all those values of $s$ in the $s$-plane for which the Laplace transform of a time-function is defined and is convergent is called the Region of Convergence (ROC) of the Laplace Transform. *Obviously the ROC of Laplace transform of a right-side function is the region to the right of Re(s) =* $\alpha$ *line.* This is a vertical straight line parallel to $j\omega$ axis and crossing $\sigma$-axis at $\alpha$ .

$V(s)$ is actually $V_1(j\omega)$ with a particular value of $\sigma$ chosen. Therefore, the time-function $v(t)$ can be extracted from $V(s)$ by inverse Fourier transform.

$$v_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_1(j\omega)e^{j\omega t}d\omega \text{ and } v_1(t) = v(t)e^{-\sigma t} \text{ with a particular value of } \sigma$$

$$\therefore v(t) = e^{\sigma t}v_1(t) = e^{\sigma t} \times \frac{1}{2\pi}\int_{-\infty}^{\infty} V_1(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} V(s)e^{\sigma t}e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(s)e^{(\sigma+j\omega)t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(s)e^{(\sigma+j\omega)t}d\omega$$

But $s = \sigma + j\omega \Rightarrow ds = d\sigma + jd\omega = jd\omega$ when the integral is evaluated with a particular value of $\sigma$

$$\therefore v(t) = \frac{1}{j2\pi}\int_{\text{on Re}(s)=\sigma\text{ line}} V(s)e^{st}ds$$

The Laplace transform defined this way returns the right-side of the underlying function $f(t)$ on inversion. The left-side returned will be zero. In this sense this Laplace transform may be termed as a *unilateral Laplace transform*. We deal with only unilateral Laplace transform in this chapter.

Note that the evaluation of inversion integral has to be performed on a line parallel to $j\omega$-axis in $s$-plane with the line crossing the $\sigma$-axis within the region of convergence of the Laplace transform.

*Let v(t) be a right-sided function that is bounded by* $Me^{\alpha t}$ *with some finite value of M and* $\alpha$*. Then the Laplace transform pair is defined as*

$$V(s) = \int_{0^-}^{\infty} v(t)e^{-st}dt \; - \text{The Analysis Equation} \qquad (15.2\text{-}1)$$

$$v(t) = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} V(s)e^{st}ds \; \text{- The Synthesis Equation} \qquad (15.2\text{-}2)$$

*where* $s = \sigma + j\omega$ *is the complex frequency variable standing for the complex exponential function* $e^{st}$ *with* $\sigma$ *value* $\geq \alpha$ *. The ROC of V(s) is the entire plane to the right of Re(s) =* $\alpha$ *line.*

If $j\omega$-axis is part of the ROC of a Laplace transform, then, substituting $s = j\omega$ in the Laplace transform will give the Fourier transform of the corresponding time-function.. This is so since if $j\omega$-axis is part of the ROC of $V(s)$, the function $v(t)$ will be absolutely integrable, its Fourier transform will exist and its Fourier transform will then be given by its Laplace transform evaluated on $j\omega$-axis. If $j\omega$-axis is not a part of the ROC of $V(s)$, the function $v(t)$ *will not be* absolutely integrable, its Fourier transform *may* or *may not* exist and if its Fourier transform exists it *will not be* the same as its Laplace transform evaluated on $j\omega$-axis.

The Laplace Transform pair

The *analysis equation* describes the decomposition of *v*(*t*) in terms of complex exponential functions.

The *synthesis equation* describes the construction of *v*(*t*) from its complex exponential function components.

That a time-function *v*(*t*) has a Laplace transform *V*(*s*) does not necessarily imply that it has a Fourier transform *V*($j\omega$) as well.

That a time-function *v*(*t*) has a Laplace transform *V*(*s*) which does not include $j\omega$-axis in its ROC does not necessarily imply that *v*(*t*) does not have a Fourier transform.

## Interpretation of Lapalce Transform

Laplace transform of a time-domain signal $v(t)$ can be interpreted in a manner similar to the interpretation of Fourier transform we carried out in Section 14.2 in Chapter 14.

The Laplace transform $V(s)$ is a complex amplitude density function. Eqn. 15.2-2 makes it clear that Laplace transform expresses the given time-function as a sum of infinitely many complex exponential functions of infinitesimal complex amplitudes. Thus Laplace transform is an expansion of $v(t)$ in terms of complex exponential functions. Fourier transform is an expansion of time-function in terms of a special class of complex exponential functions – the ones that are represented as points on $j\omega$-axis. Therefore, a Fourier transform can be evaluated only on $j\omega$-axis. But the entire ROC is available for evaluating Laplace transform. We will consider an example to clarify this matter.

### Example : 15.2-1

Find the Laplace transform of *v*(*t*) = *u*(*t*).

### Solution

$$V(s) = \int_{0^-}^{\infty} e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_{0^-}^{\infty} = \frac{1}{s} \text{for Re}(s) > 0 \;.$$

Therefore $V(s)=1/s$ with ROC of Re($s$) > 0.

Thus the inversion integral can be evaluated on any vertical straight-line on the right-half in $s$-plane. But does not that mean that a steady function like $u(t)$ is being synthesized from oscillations that grow with time? It means precisely that. The synthesis equation Eqn. 15.2-2 reveals that infinite growing complex exponential functions of infinitesimal amplitudes, which start at -∞ and go up to +∞ in time, participate in making the transient time-function $u(t)$. The contribution from a band of complex frequencies around a complex frequency value $s$ is approximately $V(s) \times \Delta s \times e^{st}$ where $\Delta s$ is the width of complex frequency band. A similar contribution comes from the band located around $s^*$. These two contributions together will form a growing sinusoidal function as shown next.

$$
\begin{aligned}
&= \frac{1}{\sigma + j\omega} \times \Delta\omega \times e^{(\sigma + j\omega)t} + \frac{1}{\sigma - j\omega} \times \Delta\omega \times e^{(\sigma - j\omega)t} \\
&= \frac{e^{\sigma t}\,[2\sigma\cos\omega t + 2\omega\sin\omega t]}{\sigma^2 + \omega^2}\Delta\omega
\end{aligned}
$$

Thus, similarly located bands in the two half-sections of the vertical line on which the inversion integral is being evaluated result in a real valued contribution as shown above. Now the inversion integral for 1/*s* can be written as

$$
\begin{aligned}
v(t) &= \frac{1}{j2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} \frac{1}{s} e^{st} ds \\
&= \frac{1}{j2\pi}\int_{0}^{\infty} \frac{e^{\sigma t}[2\sigma\cos\omega t + 2\omega\sin\omega t]}{\sigma^2 + \omega^2}(jd\omega) \qquad (15.2\text{-}3) \\
&= \frac{1}{2\pi}\int_{0}^{\infty} \frac{e^{\sigma t}[2\sigma\cos\omega t + 2\omega\sin\omega t]}{\sigma^2 + \omega^2} d\omega
\end{aligned}
$$

Thus, infinitely many exponentially growing sinusoids of frequencies ranging from zero to infinity, each with infinitesimal amplitude, interfere with each other constructively and destructively from $t$ = -∞ to $t$ = +∞ to synthesize the unit step waveform. Moreover, the exponentially growing sinusoids that participate in this waveform construction process are not unique. The value of $\sigma$ can be any number > 0. Therefore, each vertical line located in the right-half of *s*-plane yields a distinct set of infinitely many exponentially growing sinusoids which can construct the unit step waveform.

That infinitely many exponentially growing sinusoids interfere with each other to produce a clean zero for all $t < 0$ and a clean 1 for all $t > 0$ is indeed counter-intuitive and quite surprising when heard first. Maybe we need a little convincing on that. The inversion integral in Eqn. 15.2-3 was evaluated using a short computer program for various values of $\sigma$ and over finite length sections on the vertical line. In effect, the program calculated the partial integral of the form

$$
v(t) \approx \frac{1}{2\pi}\int_{0}^{\omega_0} \frac{e^{\sigma t}[2\sigma\cos\omega t + 2\omega\sin\omega t]}{\sigma^2 + \omega^2} d\omega \quad \text{for various values of } \sigma \text{ and } \omega_0.
$$

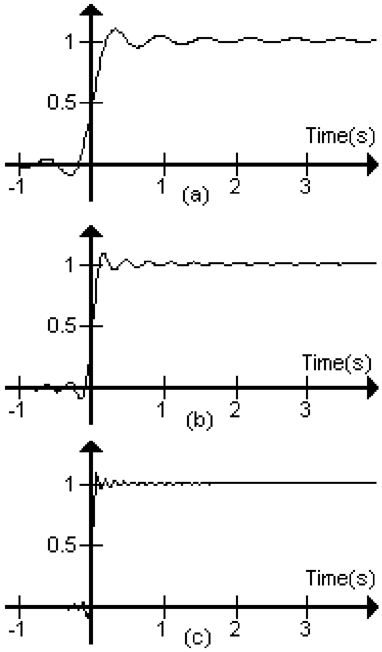

Fig. 15.2-1 Partial Inversion Integral for Unit Step Function for $\sigma$ =0.1 and (a)$\omega_0$ = 10 (b)$\omega_0$ = 20 (c)$\omega_0$ = 50

Fig. 15.2-1 shows the resulting waveforms for $\sigma$ = 0.1 neper/sec and $\omega_0$ = 10, 20 and 50 rad/sec.

Even a small range of 10 rad/sec shows the tendency of the integral to approach step waveform. With $\omega_0$ = 50 rad/sec the integral has more or less yielded step waveform at least in the range –1 sec to 4 sec. We also observe the familiar Gibbs oscillations at discontinuities. This is expected since Laplace transform after all is a kind of generalised Fourier transform. Moreover, observe that the inversion integral returns 0.5 at $t = 0$. That is the value assured by Dirichlet. This too was expected since Laplace transform is a Fourier transform in disguise.

Fig. 15.2-2 shows the results of partial evaluation of inversion integral for $\sigma$ = 1 neper/sec and $\omega_0$ = 10, 20 and 50 rad/sec.

This set of simulation result shows that we have to include more and more components in the partial integral to converge to unit step waveshape in a given time-interval as we let the components grow at a faster rate, *i.e.*, for higher values of $\sigma$. And, keeping $\sigma$ at a fixed value, we would need to include more and more frequency components when we increase the time-range over which we want convergence. However, we have infinite components at our disposal and it will be possible to include enough of them to recover the $u(t)$ shape up to any finite $t$ however large it may be.

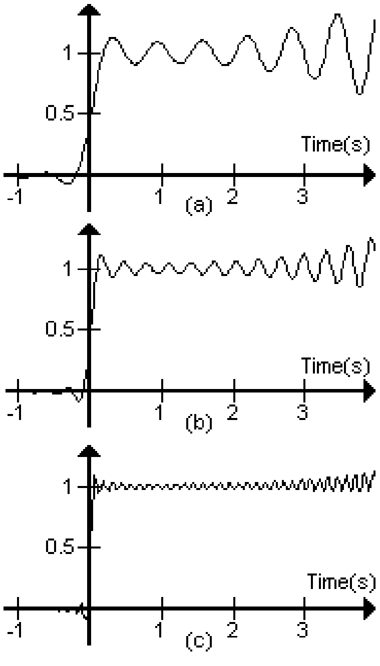

Fig. 15.2-2 Partial Inversion Integral for Unit Step Function for $\sigma$ =1 and (a) $\omega_0$ = 10 (b) $\omega_0$ = 20 (c) $\omega_0$ = 50

Therefore, Laplace transform expands a transient right-sided time-function in terms of infinitely many complex exponential functions of infinitesimal amplitudes. The ROC of such a Laplace transform will include right-half of *s*-plane and hence the time-domain waveform gets constructed by growing complex exponential functions though it appears counter-intuitive.