The switch voltage waveform is shown in Fig. 11.10-4.

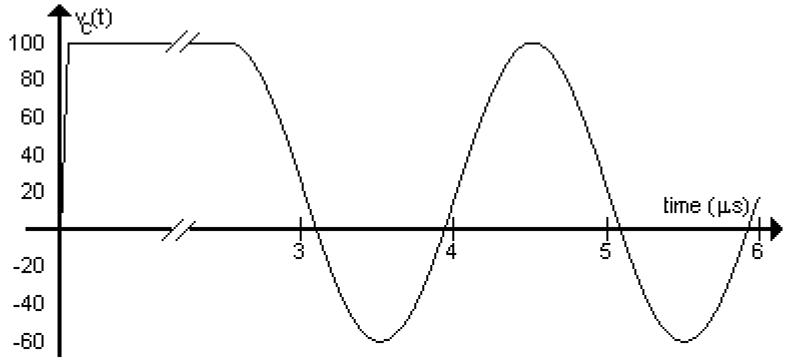

Fig. 11.10-4 Switch Voltage Waveform in Example : 11.10-4

The switch voltage will oscillate with a frequency of ~ 1MHz between 100 V and –60 V forever unless there is some damping in the circuit. There will be an accompanying current oscillation of 0.253 amps amplitude. Here too, the circuit current does not get switched off unless there is some damping in the circuit.

To find the additional capacitance needed across the switch to limit the peak to 100V

Let C_t be the total capacitance across the switch in nF units. Then,

$$v_C(t) = 20 + A_1 \sin 3.162\times10^6 t + A_2 \cos 3.162\times10^6 t$$

Then,

$$i(t) = C_t \times 10^{-6} \frac{dv_C(t)}{dt}$$

$$= 3.162\times10^{-3} C_t A_1 \cos 3.162\times10^6 t - 3.162\times10^{-3} C_t A_2 \sin 3.162\times10^6 t$$

Applying $v_C(0^+) = 0\,\text{V},\ i(0^+) = 2\,\text{A}$,

$$A_2 = -20$$

$$3.162\times10^{-3} C_t A_1 = 2$$

$$\therefore A_1 = \frac{632.5}{C_t} \text{ and } A_2 = -20$$

$\sqrt{A_1^2 + A_2^2}$ must be ≤ 80. Therefore C_t ≥ 8.17 nF. Hence the additional capacitance needed across the switch is ≥ 7.17 nF.

An inductor carrying current has certain amount of energy stored in it. It is necessary to dissipate this stored energy or reabsorb it before the current in an inductive circuit can be switched off.

The capacitance that appears across the open switch is forced to absorb this inductive energy in a switching context. Example : 11.10-4 makes it clear that it may be necessary to shunt the switch with an additional capacitor in order to limit the voltage that appears across the open switch. It also points to the need to employ a dissipating element across the switch to damp the oscillation that follows a switching operation.

A series combination of judiciously chosen capacitor and resistor with the resistor shunted by a diode is commonly employed across semiconductor switches used in inductive circuits in practice.

11.11 Frequency Response of Series *RLC* Circuit

Consider a Series *RLC* Circuit excited by a voltage source $v_S(t)$ = 1 sinω t $u(t)$ volts with a set of specified initial conditions. We continue to use $v_C(t)$, the voltage across capacitor , as the describing variable for the circuit.

The total response of the circuit is, as usual, given by the sum of zero-input response and zero-state response. Zero-input response contains only transient terms that are exponentially damped sinusoid in nature. The zero-state response contains transient terms as well as the forced response term. The transient terms in zero-state response will be of the same nature as that of transient terms in zero-input response. In short, the total response will contain natural response terms of exponentially damped sinusoidal nature and forced response term. We expect the natural response terms to vanish when time increases without limit. Only the forced response term acts in the long run, and of course, it is termed steady-state response too.

We are interested in the sinusoidal steady-state response of Series *RLC* Circuit in this section. In particular, we look at variation of the ratio of output amplitude to input amplitude and the phase angle by which output leads the input with frequency under sinusoidal steady-state – that is, we look at frequency response of Series *RLC* Circuit in detail.

Sinusoidal Forced-Response from Differential Equation

The differential equation governing $v_C(t)$, the capacitor voltage , in a Series *RLC* Circuit excited by a voltage source $v_S(t)$ was derived earlier in this chapter.

$$\frac{d^2 v_C(t)}{dt^2} + 2\xi\omega_n \frac{dv_C(t)}{dt} + \omega_n^2 v_C(t) = \omega_n^2 v_S(t)$$
$$\text{where } \omega_n = \sqrt{\frac{1}{LC}} \text{ , } \xi = \frac{R}{2\sqrt{\frac{L}{C}}} \quad (11.11\text{-}1)$$

The steady-state component of response when $v_S(t) = 1 \sin\omega t\, u(t)$ is obtained by 'method of undetermined coefficients'. We assume a trial solution of the form $v_C(t) = A \sin\omega t + B \cos\omega t$ and determine the values of A and B by substituting the assumed solution in the differential equation and equating coefficients of $\sin\omega t$ and $\cos\omega t$ on both sides of the resulting equation.

Sinusoidal steady-state response by method of undetermined coefficients

Trial Solution, $v_C(t) = A\sin\omega t + B\cos\omega t$

Substituting in the differential equation and collecting terms,

$$\left[(\omega_n^2 - \omega^2)A - 2\xi\omega_n B\right]\sin\omega t + \left[(\omega_n^2 - \omega^2)B + 2\xi\omega_n A\right]\cos\omega t = \omega_n^2 \sin\omega t$$

The only way this equation can remain true independent of t is by the coefficients of $\sin\omega t$ (and $\cos\omega t$) becoming equal on both sides of the equation. Therefore,

$$(\omega_n^2 - \omega^2)A - 2\xi\omega_n B = \omega_n^2$$
$$(\omega_n^2 - \omega^2)B + 2\xi\omega_n A = 0 \quad .$$

Solving these two equations simultaneously we get,

$$A = \frac{\omega_n^2(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2} \text{ , } B = \frac{-2\xi\omega_n^2(\omega_n\omega)}{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}$$

Substituting these in the assumed solution and simplifying the solution to a 'single sinusoid with phase' form we get,

$$v_C(t) = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}} \sin(\omega t + \phi_C)$$
$$\text{where } \phi_C = -\tan^{-1}\frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \quad (11.11\text{-}2)$$

Therefore, the magnitude part of frequency response function for $v_C(t)$ is $\dfrac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}}$ and the phase part of frequency response function is $-\tan^{-1}\dfrac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2}$ radians.

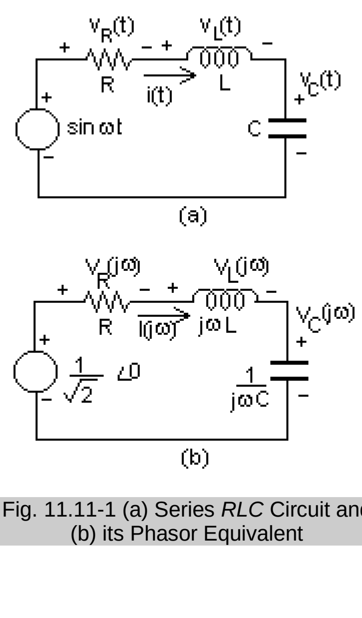

Fig. 11.11-1 (a) Series *RLC* Circuit and (b) its Phasor Equivalent

Frequency Response from Phasor Equivalent Circuit

We remember at this point that phasor analysis is another way to arrive at the sinusoidal steady-state response of linear circuits. We verify the result in Eqn. 11.11-2 by employing phasor analysis. The Series *RLC* Circuit with unit amplitude sinusoidal excitation and its phasor equivalent circuit are shown in Fig. 11.11-1.

The frequency response function for any circuit variable is the ratio of output phasor to the input phasor. This ratio will be a complex ratio and its magnitude part will give the amplitude ratio and its angle part will give the phase angle by which the output sine wave *leads* the input sine wave under steady-state condition. We obtain three phasor ratios for the voltage variables in this circuit by employing voltage division principle in a series circuit.

$$\frac{V_C(j\omega)}{V_S(j\omega)} = \frac{1/j\omega C}{R + j\omega L + 1/j\omega C} = \frac{1}{1-\omega^2 LC + j\omega RC}$$
$$= \frac{1/LC}{1/LC - \omega^2 + j\omega R/L} = \frac{\omega_n{}^2}{(\omega_n{}^2 - \omega^2) + j2\xi\omega_n\omega}$$

This ratio can be written in polar form as

$$\frac{V_C(j\omega)}{V_S(j\omega)} = \frac{\omega_n{}^2}{\sqrt{(\omega_n{}^2-\omega^2)^2 + 4\xi^2\omega_n{}^2\omega^2}}\angle\phi_C$$
$$\text{where } \phi_C = -\tan^{-1}\frac{2\xi\omega_n\omega}{\omega_n{}^2-\omega^2} \text{ rad} \qquad (11.11\text{-}3)$$

Frequency response of capacitor voltage in a Series *RLC* circuit

We see that the frequency response obtained by solving the differential equation is the same as the one obtained by employing phasor equivalent circuit as expected. Similar evaluation of phasor ratios leads to the frequency response functions for the remaining two voltage variables in the circuit. They are,

$$\frac{V_R(j\omega)}{V_S(j\omega)} = \frac{j2\xi\omega\omega_n}{(\omega_n{}^2-\omega^2) + j2\xi\omega_n\omega}$$
$$= \frac{2\xi\omega\omega_n}{\sqrt{(\omega_n{}^2-\omega^2)^2 + 4\xi^2\omega_n{}^2\omega^2}}\angle\phi_R \qquad (11.11\text{-}4)$$
$$\text{where } \phi_R = \frac{\pi}{2} - \tan^{-1}\frac{2\xi\omega_n\omega}{\omega_n{}^2-\omega^2} \text{ rad}$$

Frequency response of resistor voltage in a Series *RLC* circuit

$$\frac{V_L(j\omega)}{V_S(j\omega)} = \frac{(j\omega)^2}{(\omega_n{}^2-\omega^2) + j2\xi\omega_n\omega}$$
$$= \frac{\omega^2}{\sqrt{(\omega_n{}^2-\omega^2)^2 + 4\xi^2\omega_n{}^2\omega^2}}\angle\phi_L \qquad (11.11\text{-}5)$$
$$\text{where } \phi_L = \pi - \tan^{-1}\frac{2\xi\omega_n\omega}{\omega_n{}^2-\omega^2} \text{ rad}$$

Frequency response of inductor voltage in a Series *RLC* circuit

The remaining variable, $i(t)$, is directly related to $v_R(t)$ and hence its frequency response need not be obtained separately.

Qualitative Discussion on Frequency Response of Series *RLC* Circuit

The ratio of voltage appearing across an element in a series circuit to the source voltage is equal to the ratio between the phasor impedance of that element to the sum of all the impedances in series. The impedance of an inductor increases linearly with angular frequency and impedance of a capacitor decreases in inverse proportion to angular frequency. We use these basic principles to discuss the shape of frequency response plots for the three possible outputs in the Series *RLC* Circuit.

<u>The Voltage Across Resistor – The Band-pass Output</u>

At zero frequency (that is, for dc steady-state) the inductor appears as a short and capacitor appears as open. Therefore, the magnitude part of frequency response is zero for $v_R(t)$, zero for $v_L(t)$ and 1 for $v_C(t)$ at this frequency.

The inductor appears as impedance of infinite magnitude and capacitor appears as impedance of zero magnitude as ω increases without limit. Therefore, all the high frequency voltage will appear across the inductor. Thus as $\omega \to \infty$, the magnitude part of frequency response is zero for $v_R(t)$ and $i(t)$, zero for $v_C(t)$ and 1 for $v_L(t)$.

The sign of impedance of inductor is positive and the sign of impedance of capacitor is negative for any ω. Thus they tend to cancel each other partially in the sum at all frequencies. The cancellation is 100% at one particular frequency. The value of frequency at which this happens is when $\omega L = 1/\omega C \Rightarrow \omega = 1/\sqrt{(LC)}$. But this frequency was named as *undamped natural frequency* earlier. Thus, we conclude that, the reactance part of the total series impedance of a Series *RLC* Circuit goes to zero at ω_n and the circuit appears purely resistive under steady-state conditions at that frequency. Therefore, the current in the circuit at that frequency will be $v_S(t)/R$ and will be in phase with the input voltage. The power factor of the circuit will be unity at that frequency.

The cancellation between the inductive reactance and capacitive reactance is only partial at all other frequencies. Hence, the magnitude of total series impedance of the circuit at any frequency other than ω_n will be more than R and amplitude of current will be less than $1/R$ amps (assuming unit amplitude excitation) at all other frequencies. *Thus, in a Series RLC Circuit, the impedance is a minimum and amplitude of current (and hence amplitude of voltage across the resistor) is a maximum at* ω_n. Moreover, the current in the circuit will be at unity power factor at that frequency. This condition in the Series *RLC* Circuit is called the *resonance* condition and the frequency at which this happens is called the *resonant frequency*. Obviously, in a Series *RLC* Circuit, the *resonant frequency* and the *undamped natural frequency* are the same.

In general, in a circuit excited by a single sinusoidal voltage source (current source) across a pair of terminals, *resonance* is the condition under which the current drawn at the terminals (voltage appearing across the terminals) is in phase with the source voltage (current).

Equivalently, *resonance* is the condition under which the input impedance (admittance) offered to the sinusoidal source is resistive.

The amplitude of voltage appearing across the resistor in a Series *RLC* Circuit under resonance condition is same as the amplitude of input. Therefore, the magnitude of frequency response for $v_R(t)$ begins with zero at zero frequency, goes to unity at ω_n and tapers down to zero as $\omega \to \infty$. The total reactance in a Series *RLC* Circuit is capacitive for $\omega < \omega_n$ and it is inductive for $\omega > \omega_n$. Therefore the voltage across resistor leads the input voltage for frequencies lower than resonant frequency and lags the input voltage for frequencies higher than resonant frequency. Thus, the phase of frequency response of $v_R(t)$ starts at 90^0 at $\omega = 0$, becomes zero at $\omega = \omega_n$ and decreases to -90^0 as $\omega \to \infty$.

Eqn. 11.11-4 confirms all these conclusions. The shape of magnitude response and phase response for the voltage across resistor is plotted against ω/ω_n ratio for various damping factors in Fig. 11.11-2.

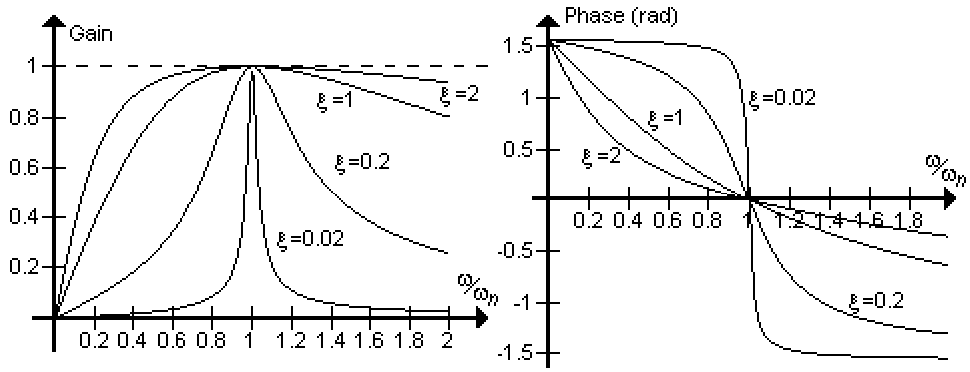

Fig. 11.11-2 Magnitude and Phase Plots for Frequency Response of Resistor Voltage in a Series *RLC* Circuit

The magnitude curve is found to become narrower as the damping ratio is reduced. The resistor voltage in a Series *RLC* Circuit exhibits the so-called *band-pass* characteristic. A frequency response is said to be of *band-pass* nature when it attenuates low frequency sinusoids and high frequency sinusoids considerably and passes on mid-frequency sinusoids preferentially. Resistor voltage in a Series *RLC* Circuit is a band-pass output for all values of damping factor – that is, even an overdamped Series *RLC* Circuit behaves as a band-pass filter if the output is taken across the resistor. However, the *band-pass* characteristic becomes sharper and sharper when the damping in the circuit is reduced. That is, the circuit becomes highly *frequency-selective* as ξ approaches zero.

Series *RLC* Circuit with output taken across *R* is a *narrow band-pass filter* for low values of ξ ($\xi < 0.1$ or $Q > 5$) and it is a *wide band-pass filter* for high value of ξ ($\xi > 0.25$ or $Q < 2$).

Another point of great significance is that the output in this band-pass filter is *in-phase* with the input at a frequency that is at the center of the band - *i.e.*, at ω_n. Output signal undergoes a phase change by about 180^0 when its frequency varies in a small band around ω_n if ξ is very small. See the phase curve for $\xi = 0.02$ in Fig. 11.11-2. This kind of rapid variation of phase of output over a small frequency range has considerable negative implications in designing control systems for systems that involve *RLC* circuits.

The Voltage Across Capacitor – The Low-pass Output

The magnitude response for this output starts at unity at zero frequency and goes to zero as $\omega \to \infty$. In between it may be a monotonically decreasing function or it may attain a maximum depending on the damping present in the circuit. The frequency response function shown in Eqn. 11.11-3 is plotted in Fig. 11.11-3 for the various values of ξ.

A Series *RLC* circuit with voltage input and output taken across the capacitor can be designed to function as a good low-pass filter.

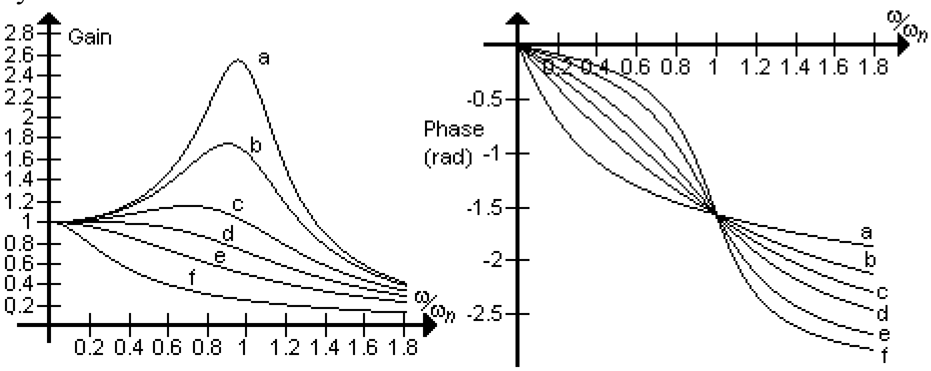

Fig. 11.11-3 Magnitude and Phase Plots for Capacitor Voltage Frequency Response in a Series *RLC* Circuit (a) ξ=0.2 (b) ξ=0.3 (c) ξ=0.5 (d) ξ=0.7 (e) ξ=1 (f) ξ=2

This output is essentially a *low-pass* output. However, it is a bad low-pass filter if it is too under-damped or too over-damped. This is so because we want the magnitude response of a low-pass filter to be reasonably flat till a particular value of ω and then fall to zero more or less rapidly. Only curves (c) and (d) in Fig. 11.11-3 look good from this point of view. In fact the value of ξ used in filter design is 0.7 typically and this corresponds to curve (d).

Observe that for $\xi < 0.7$ the magnitude response exhibits a peak. The ratio of this peak gain value to dc gain value is called *resonant peak factor*. The frequency at which the resonant peak in gain occurs *is not* the frequency at which resonance occurs in the circuit. Resonance occurs in the circuit at ω_n and the voltage across the resistor goes to a maximum value at that frequency. But the frequency at which the capacitor voltage goes to a maximum for fixed input amplitude is different from ω_n and it depends on ξ too. This frequency is less than ω_n and shifts more to the left with increase in ξ. The expression for magnitude response Eqn. 11.11-3 may be differentiated with respect to ω and set to zero to find the frequency at which maximum takes place (if at all there is a maximum) and the value of the maximum. The results will be

$$\omega_{cp} = \omega_n \sqrt{1-2\xi^2} \text{ and } R_{cp} = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{Q}{\sqrt{1-\frac{1}{4Q^2}}}$$

Resonant peak frequency and resonant peak factor in the low-pass output in a Series *RLC* circuit

where ω_{cp} is the frequency at which gain maximum takes place and R_{cp} is the resonant peak factor. The expressions reveal that there is a resonant peak only for $\xi < 1/\sqrt{2} \approx 0.7$.

The gain for capacitor voltage at ω_n is $1/(2\xi)$ (=Q) as shown by Eqn. 11.11-3 with $\omega = \omega_n$. This predicts large amplitude voltage across the capacitor when the damping factor is small. For example, the amplitude of voltage across the capacitor is 50 volts when a 1 volt sinusoid is applied to the circuit at resonant frequency if the ξ factor is 0.01 (equivalently, Q factor is 50). How does this voltage amplification take place?

The voltage gain available across the capacitor in a Series *RLC* circuit under sinusoidal steady-state exhibits a peak for all $\xi > 0.707$.

The gain available at resonant frequency is equal to the quality factor *Q* of the circuit. However, this is not the peak gain.

The peak gain is >*Q* and is available at a frequency less than resonance frequency.

We have seen that the series circuit impedance is resistive and a minimum at ω_n. The reactance of L and C cancel each other at that frequency. Hence R decides the amplitude of current and the reactance of C multiplies this current amplitude to convert it into voltage amplitude. Therefore the ratio of amplitude of capacitor voltage to amplitude of input voltage at ω_n must be $1/\omega_n RC$. Similarly the ratio of amplitude of inductor voltage to amplitude of input voltage at ω_n must be $\omega_n L/R$. But, both these ratios are equal to the Q factor of the circuit as seen below.

$$\frac{1}{\omega_n RC} = \frac{\sqrt{LC}}{RC} = \frac{\sqrt{L/C}}{R} = \frac{1}{2\xi} = Q$$

$$\frac{\omega_n L}{R} = \frac{L}{\sqrt{LC}R} = \frac{\sqrt{L/C}}{R} = \frac{1}{2\xi} = Q$$

Thus the voltage amplification factor for capacitor voltage and inductor voltage in a Series *RLC* Circuit at resonant frequency is its Q factor.

Resonant amplification of voltage in a Series *RLC* Circuit is one method used in High Voltage Engineering to generate high ac voltages from low voltage sources. Such amplification becomes possible because at resonant frequency the voltage across L will be of same amplitude and opposite phase as that of the voltage across C and therefore they cancel each other without absorbing any portion of input voltage to sustain them.

<u>The Voltage Across Inductor – The High-pass Output</u>

The magnitude response for this output starts at zero at zero frequency and goes to unity as $\omega \to \infty$. In between it may be a monotonically increasing function or it may attain a maximum depending on the damping present in the circuit. The frequency response function shown in Eqn. 11.11-5 is plotted in Fig. 11.11-4 for various values of ξ.

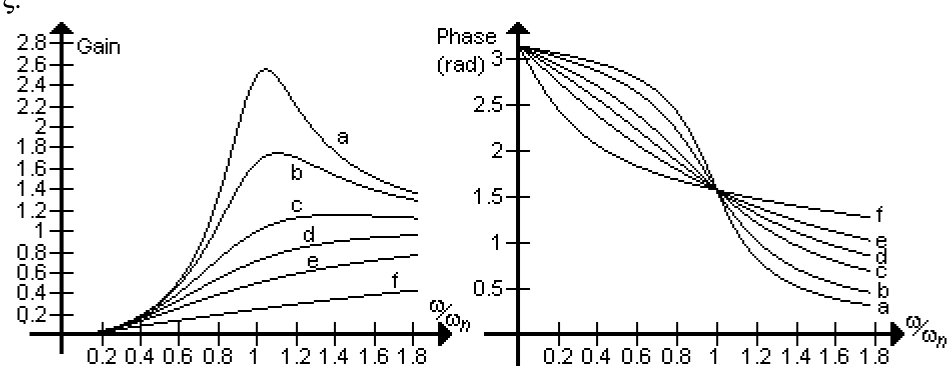

Fig. 11.11-4 Magnitude and Phase Plots for Inductor Voltage Frequency Response in a Series *RLC* Circuit (a) ξ=0.2 (b) ξ=0.3 (c) ξ=0.5 (d) ξ=0.7 (e) ξ=1 (f) ξ=2

The plots as well as our qualitative discussion reveal that the inductor voltage in a Series *RLC* Circuit has *high-pass nature* under sinusoidal steady-state. But, it is a bad high-pass filter if the Series *RLC* Circuit is only lightly damped. A good high-pass filter is expected to have near-zero gain at low frequencies till a particular frequency and a gain that rapidly rises to unity and remains there after that frequency. The prominent resonant peaks in the gain evident in magnitude response plot in Fig. 11.11-4 for low ξ values are not acceptable in a good high-pass filter. Such resonant peaks will lead to considerable distortion even for signals that do not have any low frequency sinusoids in them.

The voltage across inductor in Series *RLC* Circuit *leads* the input voltage by a phase angle that varies from 180^0 to 0^0 with ω. The phase angle is 90^0 at ω_n.

The angular frequency at which the voltage across the inductor peaks for given input voltage amplitude *is not* ω_n. It is always greater than ω_n and moves further to the right of ω_n as ξ increases. Exact expression for the angular frequency at which the magnitude response peaks and the value of the peak (if such resonant peaking occurs at all) can be obtained by differentiating the magnitude expression in Eqn. 11.11-5 with respect to ω and setting the derivative to zero. The result will be

$$\omega_{lp} = \frac{\omega_n}{\sqrt{1-2\xi^2}} \text{ and } R_{lp} = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{Q}{\sqrt{1-1/4Q^2}}$$

where ω_{lp} is the frequency at which gain maximum takes place and R_{lp} is the resonant peak factor. The expressions reveal that there is a resonant peak only for $\xi < 1/\sqrt{2} \approx 0.7$.

We observe that $\omega_{cp}\omega_{lp} = \omega_n^2$ and $R_{cp} = R_{lp}$. Thus the geometric mean of resonant peak frequencies at high-pass and low-pass outputs is equal to the resonant peak frequency at the band-pass output.

The magnitude response for all the three outputs are shown in Fig. 11.11-5 for $\xi = 0.3$. The points brought out in the discussion on resonant peaks and the frequencies at which resonant peaks occur are illustrated in this plot.

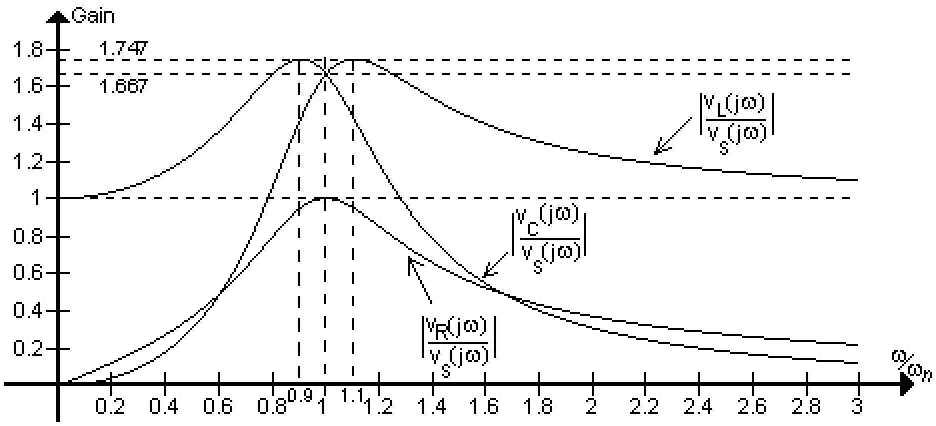

Fig. 11.11-5 Magnitude Response for Low-Pass, Band-Pass and High-Pass Outputs in a Series *RLC* Circuit with Damping Factor = 0.3

The capacitor voltage attains a peak amplitude of 1.747 volts at ~$0.9\omega_n$ and the inductor voltage attains a peak amplitude of 1.747 volts at ~ $1.1\omega_n$ for an input amplitude of 1 volt. Both attain amplitude of 1.667 volts at resonant frequency ω_n. The *Q* factor of the circuit is 1.67.

A More Detailed Look at the Band-pass Output of Series *RLC* Circuit

The frequency response at the band-pass output of Series *RLC* Circuit (*i.e.,* the voltage across the resistor) was obtained as

$$\frac{V_R(j\omega)}{V_S(j\omega)} = \frac{j2\xi\omega\omega_n}{(\omega_n^2-\omega^2)+j2\xi\omega_n\omega} = \frac{2\xi\omega\omega_n}{\sqrt{(\omega_n^2-\omega^2)^2+4\xi^2\omega_n^2\omega^2}}\angle\phi_R \qquad (11.11\text{-}6)$$

$$\text{where } \phi_R = \frac{\pi}{2} - \tan^{-1}\frac{2\xi\omega_n\omega}{\omega_n^2-\omega^2} \text{ rad}$$

Frequency response function for band-pass output in a Series *RLC* circuit

It was shown that this output has a maximum gain of unity and that occurs at $\omega = \omega_n$. We wish to develop a measure for the *frequency selectivity* exhibited by this output in this sub-section.

The resistor gets all of the input voltage if input frequency is resonant frequency. Therefore the resistor receives maximum power for fixed amplitude input if the input frequency is resonant frequency. The power dissipated at all other frequencies will be less than this value. There will be one or more frequencies at which the power dissipated in the resistor is exactly 50% of the power that will be dissipated if input is at resonant frequency (assuming the input amplitude is kept fixed). These frequencies have been used traditionally to define measures of frequency selectivity in band-pass circuits. They are called *half-power frequencies* for obvious reason.

'Half-power frequencies' defined and explained as a measure of frequency selectivity of band-pass circuits

Power dissipated in a resistor becomes 50% when voltage developed across it becomes $1/\sqrt{2} \approx 0.707$. Therefore, *half-power frequencies* are the angular frequencies at which the magnitude response of the circuit output become 70.7% of some reference gain value. The reference gain value in the case of low-pass circuit is the dc gain, in the case of band-pass circuit it is the maximum gain and in the case of high-pass circuit it is the gain as $\omega \to \infty$.

The frequency at which the gain of a band-pass circuit reaches maximum is termed *center frequency* in filter studies. A typical band-pass response is shown in Fig. 11.11-6. The half power frequencies ω_1 and ω_2 are marked in the magnitude plot.

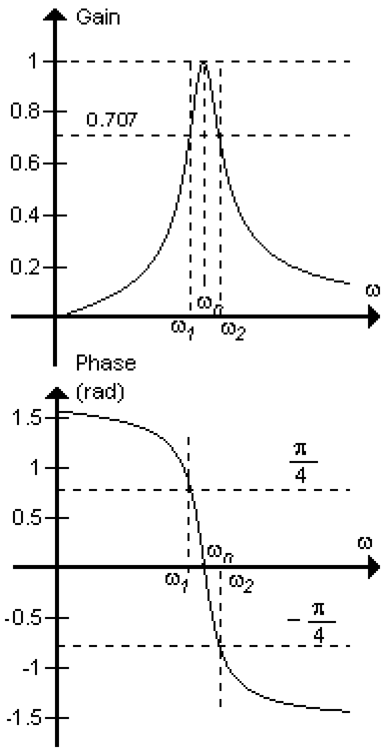

Fig. 11.11-6 A Typical Band-pass Circuit Frequency Response

The ratio of center frequency to bandwidth of a narrow band-pass filter is equal to its quality factor.

The half-power frequencies of such a filter are at approximately equal distance on either side of the center frequency.

The difference between the two half-power frequencies is called the *bandwidth* of the band-pass circuit. We develop an expression for bandwidth of a *narrow band-pass* circuit below and develop interesting insight into the relation between the bandwidth and quality factor of the circuit.

$$\frac{2\xi\omega\omega_n}{\sqrt{(\omega_n{}^2-\omega^2)^2+4\xi^2\omega_n{}^2\omega^2}}=\frac{1}{\sqrt{2}}$$

$$\therefore \frac{(2\xi\omega\omega_n)^2}{(\omega_n{}^2-\omega^2)^2+4\xi^2\omega_n{}^2\omega^2}=\frac{1}{2}$$

$$\text{Let } x=\frac{\omega}{\omega_n}.\text{Then, } \frac{4\xi^2x^2}{(1-x^2)^2+4\xi^2x^2}=\frac{1}{2}$$

$$\therefore (1-x^2)^2-4\xi^2x^2=0 \Rightarrow (1-x^2)=\pm 2\xi x$$

$$\therefore x^2 \pm 2\xi x-1=0 \Rightarrow x=\mp\xi\pm\sqrt{1+\xi^2}$$

Taking only positive values for ω, $\omega_{2,1}=(\sqrt{1+\xi^2}\pm\xi)\omega_n \Rightarrow bw=2\xi\omega_n$

$$\therefore \frac{\text{Center frequency}}{\text{Bandwidth}}=\frac{1}{2\xi}=Q \qquad (11.11\text{-}7)$$

This Q factor (equivalently, the damping factor) has indeed turned out to be an important parameter for Series *RLC* Circuit. Eqn. 11.11-7 is the third interpretation for Q.

Different interpretations for Q-factor of a *RLC* circuit

We had seen earlier that, in a weakly damped Series *RLC* Circuit, the fractional loss of total stored energy in the circuit over one cycle of oscillation is given by $4\pi\xi$. Since $Q = 1/2\xi$,

$$Q=2\pi\frac{\text{Total stored energy in the source-free circuit}}{\text{Energy lost in one cycle of free response}}.$$

Second interpretation is based on the same energy ratio under sinusoidal steady-state conditions at resonant frequency. Let the circuit be at resonance with 1-Volt amplitude input. Then,

$$v_S(t)=1\sin\omega_n t,\ \therefore i(t)=\frac{1}{R}\sin\omega_n t \ \text{ and } v_C(t)=\frac{-1}{\omega_n RC}\cos\omega_n t$$

$$\text{Total stored energy}=\frac{Li(t)^2}{2}+\frac{Cv_C(t)^2}{2}=\frac{L}{2R^2}\sin^2\omega_n t+\frac{C}{2\omega_n{}^2R^2C^2}\cos^2\omega_n t$$

$$=\frac{L}{2R^2}(\because \omega_n{}^2=\frac{1}{LC})$$

$$\text{Energy dissipated in one cycle}=\frac{1}{\omega_n}\int_0^{2\pi}\mathrm{R}\times\left(\frac{1}{R}\sin\omega_n t\right)^2 d(\omega_n t)=\frac{\pi}{\omega_n R}$$

$$\therefore \frac{\text{Total stored energy}}{\text{Energy dissipated in one cycle}}=\frac{1}{2\pi}\frac{\omega_n L}{R}=\frac{1}{2\pi}\frac{\sqrt{L/C}}{R}=\frac{1}{2\pi}\frac{1}{2\xi}=\frac{Q}{2\pi}$$

$$\therefore Q=2\pi\frac{\text{Total stored energy under resonance condition}}{\text{Energy dissipated in one cycle under resonance condition}}$$