10.5 Features of *RL* Circuit Step Response

Step Response in electrical circuit analysis context implies application of the unit step function, $u(t)$, as the input with a set of *zero-valued* initial conditions specified for the circuit at $t = 0^-$. The response to this unit step application can be described in terms of a chosen circuit variable, which may be a voltage variable, a current variable or a linear combination of voltage and current variables. We had chosen the current through inductor as the response variable in the case of series *RL* circuit. The current waveform with zero initial condition (initially relaxed circuit) was shown to be

$$i_L = \frac{1}{R}(1-e^{-t/\tau})\ ;\ t \geq 0^+ \quad (10.5\text{-}1)$$

The primary objective in applying an input function to a circuit is to make some chosen output variable in the circuit behave in a desired manner. This is why the input function gets called *forcing function*. Input function is a command to the circuit to vary its response variable in a manner similar to its own time variation. Application of unit step input is equivalent to a command to the circuit to change its response variable in a step-wise manner in this sense. Similarly, when we switch on a voltage $v_S(t) = V_m \sin \omega t$ volts at $t = 0$ to any circuit, we are, in effect, commanding the circuit to make the chosen response variable follow this function in shape. A purely memory-less circuit will follow the input command with no delay. However circuits with memory elements will not.

The objective in applying a forcing function to a circuit

Inductor constitutes electrical inertia. It does not like to change its current and resists any such current change by producing a back e.m.f across it – the magnitude of this e.m.f is directly proportional to the rate at which the inductor current changes. Other elements in the circuit (usually voltage sources, switches etc.) will have to supply the voltage demanded by the inductor if the desired current change is to take place. This is the price the other elements in the circuit have to pay for demanding the lazy inductor to change its current. The price is heavier if the required change in current is to be accomplished faster.

Inductance constitutes electrical inertia

It is still more instructive to look at the 'inertia' aspect of inductor from its energy storage capability. An inductor stores energy in its magnetic field. The energy stored in the field is proportional to square of current through inductor. Thus, if we want to change the current through inductor, we have to supply energy to the inductor or absorb energy from the inductor. Note that we do not have to do any such thing if the current through inductor is at a constant level. Let us assume that we want to change the inductor current from I_1 to I_2 ($I_2 > I_1$). By the time we have done it, we would have given the inductor $0.5L(I_2^2 - I_1^2)$ Joules of energy. We can pump energy into the inductor only by pumping power into it. Therefore, a voltage has to appear across the inductor whenever its current tries to change. And energy has to be pumped into inductor at a fast rate if the current in inductor is to change fast. That means that power flow into inductor has to be increased if the inductor current is to change fast. And that is why voltage across inductor becomes higher and higher when a given amount of current–change is sought to be attained in shorter and shorter time intervals.

Inertia of inductance looked at from stored energy point of view

Consider a similar situation in translational mechanics. A mass M is forced to move against friction. Assume that the frictional force is proportional to velocity of the mass and that there is no sticking friction. Now, if we apply a constant force to the mass we know that (i) the mass reaches a final speed at which the applied force is met exactly by the frictional force acting against motion (ii) it takes some time to reach this situation. Mass M does not like to move due to its inertia – it is in the nature of objects in this world to stay put. They prefer it that way. Similarly, it is in the nature of inductor to stay put as far as its current is concerned. However, objects in this world do yield to forces eventually. In the above case, since the mass M shows a tendency to stay put even after the force has come into action, it has to absorb the entire force initially. In that process it gets accelerated. Hence, for a brief period initially, a major portion of the applied force goes to accelerate the mass and only a minor portion goes for meeting friction. This proportion will change with time and finally no force will be spent on accelerating the mass and entire force will be spent on countering friction. Hence, initially the 'inertial

Series *RL* circuit compared with a Mass moving against viscous friction nature' of mass dominates the situation and puts up a stiff fight with the force that is a command to the mass to move at a constant speed. Slowly the resistance from the mass weakens and inexorably the force subjugates the inertial nature of mass. And after sufficient time has elapsed, the applied force wins the situation; the mass yields almost completely to the force command and moves at an almost constant speed commensurate with the level of friction present in the system.

This tussle between the inherent inertial nature of systems and the compelling nature of forcing functions is a common feature in dynamic systems involving memory elements and is present in electrical circuits too. Thus, the response immediately after the application of a forcing function in a circuit will be a compromise between the inherent natural laziness of the system and the commanding nature of forcing function. The circuit expresses its dislike to *change* by spewing out a time function, which quantitatively describes its unwillingness to *change*. The forcing function wears down this natural cry from the circuit gradually and establishes its supremacy in the circuit in the long run – by forcing all circuit variables to vary as per its dictate in the long run.

The total response in the circuit is always a mixture of these two with the component from forcing function dominating almost entirely in the long run and the natural component from the circuit's inherent inertia ruling in the beginning. It should be noted at this point that it is quite possible that neither component will succeed in overpowering the other in some circuits. Such circuits are called *marginally stable circuits*. And there are circuits in which the natural component will not only refuse to yield but grow without limit as time increases; thereby overpowering the forcing function with time. Such circuits are called *unstable circuits*. We will take up such circuits in later chapters. But at present we deal with circuits that yield to the forcing function in the long run – called *stable* circuits.

Natural response and *forced response* defined and distinguished

The time function that the circuit employs to protest against *change* is called the *natural response* of the circuit and the time function that the forcing function establishes in the response variable is called the *forced response*. The *natural response* means precisely that – it encodes the basic nature of the circuit and has nothing to do with the nature of forcing function. Its shape and other features (except amplitude) are decided by the nature and number of energy storage elements in the circuit, the way these energy storage elements are connected along with resistive elements to form the circuit etc. Thus its shape depends only on the nature of elements and the topology of the circuit and does not depend on the particular shape and value of forcing function – it is *natural* to the circuit. But its magnitude will depend on initial condition and forcing function too.

The series *RL* circuit with voltage source excitation howls '*exponentially'* when forcing function commands its current to change. In fact, all stable dynamic systems described by a 'linear first order ordinary differential equation with constant coefficients' will cry out exponentially when they are asked to change. They all have a natural response of the type $Ae^{-\alpha t}$ where α, which decides the shape of response, is decided by system parameters (*R* and *L* in the present instance) and *A* is decided by initial condition and the initial value of forced response. The forcing function along with initial condition will decide the magnitude of natural response; but not its shape.

The complementary solution of the describing differential equation of a circuit yields the natural response of the circuit..

The particular integral corresponding to the applied forcing function yields the forced response.

The shape of natural response does not depend on forcing function and hence must be the same with or without forcing function. A non-zero response with a zero forcing function can exist if the circuit starts out with initial energy at $t = 0^-$. This is similar to a mass, which has been accelerated to some velocity before t=0, slowing down to zero speed after t=0 under the effect of friction. Thus, it follows that we can find out the shape of natural response by solving the equation describing the circuit response with forcing function set to zero. But that will be the homogeneous differential equation and we know that its solution is the complementary function of the equation. The complementary solution of the differential equation describing the current in the inductor in our *RL* circuit was shown to be an exponential function with negative real index earlier. Thus, we conclude that *the complementary solution of the describing differential equation of a circuit yields the natural response of the circuit whereas the particular integral corresponding to the applied forcing function yields the forced response.*

Step Response Waveforms in Series *RL* Circuit

There are only three circuit variables in a series *RL* circuit and they are $i_L(t)$,$v_R(t)$ and $v_L(t)$ as marked in Fig. 10.1-1 . The expressions for $v_R(t)$ and $v_L(t)$ may be worked out from the solution for $i_L(t)$.These expressions for zero initial condition are

$$\begin{aligned} i_L(t) &= \frac{1}{R}(1-e^{-t/\tau})\ ;\ t \geq 0^+ \\ v_R(t) &= (1-e^{-t/\tau})\ ;\ t \geq 0^+ \\ v_L(t) &= L\frac{di_L(t)}{dt} = \frac{L}{R} \times \frac{e^{-t/\tau}}{\tau} = e^{-t/\tau};\ t \geq 0^+ \end{aligned} \quad (10.5\text{-}2)$$

Step response of an initially relaxed Series *RL* circuit

We introduce normalisation in these expressions before we plot them by dividing the expressions by the final steady value or the maximum value as applicable. Similarly we define normalised time variable t_n as t/τ. This results in

$$\begin{aligned} i_{Ln}(t) &= \frac{i_L(t)}{1/R} = (1-e^{-t_n})\ ;\ t_n \geq 0^+ \\ v_{Rn}(t) &= \frac{v_R(t)}{1} = (1-e^{-t_n})\ ;\ t_n \geq 0^+ \\ v_{Ln}(t) &= \frac{v_L(t)}{1} = e^{-t_n};\ t_n \geq 0^+ \end{aligned} \quad (10.5\text{-}3)$$

Normalised step response of an initially relaxed Series *RL* circuit

where the second subscript '*n*' indicates normalised variables.

These waveforms appear in Fig. 10.5-1. The inductor current rises from zero level at $t = 0^+$ and approaches the normalised final value of 1 as t_n approaches ∞. It never touches the final value of 1 since the exponential function never becomes zero. The growth of inductor current gradually loses momentum resulting in the convex shape of current waveform. Simultaneously, the voltage across inductor decays exponentially and tends to go to zero as t_n approaches ∞.

Most of the rise in i_L and fall in v_L take place within first three units of normalised time *i.e.*, within 3τ seconds of actual time. The values of e^{-1}, e^{-2} and e^{-3} are 0.368, 0.135 and 0.05 respectively. Hence the step response current in an *initially relaxed RL* series circuit rises to 63.2% of its final value in the first τ seconds. The voltage across inductor in the same circuit falls *by* 63.2% from its value at $t = 0^+$ to reach 36.8% of its initial value in the first τ seconds. During the second τ second period, inductor current rises by another 23.3% to reach 86.5% of its final value. Correspondingly the voltage across inductor falls to 13.5% of its initial value at the end of 2τ seconds. And at the end of 3τ seconds 95% of the transient is over and the inductor current is only 5% away from its final value.

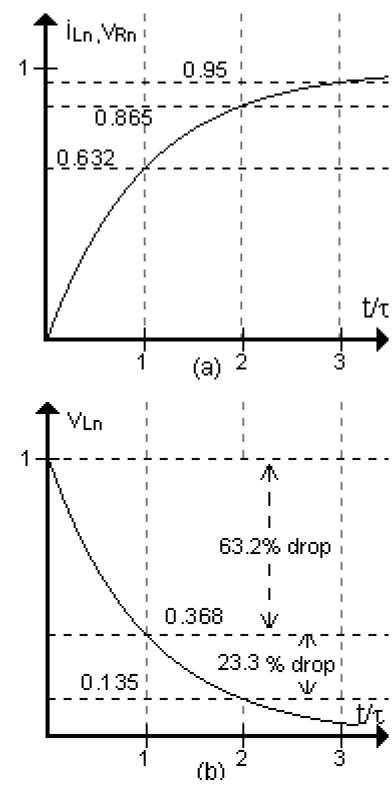

Fig. 10.5-1 Series *RL* Circuit Step Response - Normalised Waveforms

The value of inductor current is 99% and 99.33% of final value at 4.6τ seconds and 5τ seconds respectively. Hence we can consider the natural response of an initially relaxed series *RL* circuit to be practically over within 5τ seconds where $\tau = L/R$. During the first five τ periods, the response in the circuit is undergoing a *transient* phase before reaching a practically *steady* situation. This period - *i.e.*, the period during which the natural response component is not negligible- is termed as the *transient period* and a value of 5τ is usually assigned to it in the case of first order circuits. This leads to another name for natural response – it is also called the *transient response*. But we have to be careful about this name since it gives us an impression that this response component is only transient and hence it will vanish with time invariably. This is not always so. There are circuits in which natural response either persists indefinitely or increases with time. So do not expect transients to vanish with time in all cases.

In the case of initially relaxed series *RL* circuit with unit step voltage input the natural response (or transient response) term in inductor current is $-e^{-t/\tau}/R$ amps and the forced response is $1/R$ amps.

The Time Constant 'τ' of a Series *RL* Circuit

The quantity *L/R* symbolically represented by τ has turned out to be an important one for *RL* circuit by now. The unit of *L* is Volt-Amp/Sec and the unit of *R* is Volt/Amp resulting in a dimension of time for this quantity with Seconds as its unit. Hence this quantity is defined as the *Time Constant* of series *RL* circuit.

> One of the interpretations assigned to the time constant is that it is the time taken by an initially relaxed *RL* circuit to reach 63.2% of its final current value. That is not very satisfactory. After all, 63.2% is not a neat round number or a particularly significant one. 50% would have been a neat measure. In fact the time taken by a first order system step response to reach 50% of its final value is termed its *half-life* and this turns out to be $\approx 0.693\tau$ in the case of *RL* circuit.

Speaking qualitatively, we can appreciate the fact that τ is a measure of the duration taken by the circuit to reach the final current value. *RL* circuit causes delay in step response current due to the memory capability of inductor. The inductor remembers its initial state and its memory prevents it from allowing sudden changes in current through it. But how deep is its memory in time? How persistent is its memory? Time constant provides answers to these questions.

A large τ implies deeper memory and consequently increased duration will be required for the forcing function to compel the inductor to go to final current value. A larger τ implies a more persistent memory and a heightened tendency on the part of the circuit to keep its current smooth in time-domain.

Time constant has another interesting interpretation. We had noted earlier that the current in the initially relaxed *RL* circuit starts rising at the rate of $1/L$ amps/sec at $t = 0^+$. Refer to Fig. 10.5-2. Tangents to the normalised inductor current plot and the normalised inductor voltage plot are drawn at normalised time instants of 1, 2 and 3. It is clear from this figure that if the inductor current had continued to rise at the same rate of rise it had at $t = 0^+$, it would reached the final value at $t_n = 1$ unit *i.e.*, at $t = \tau$. The slope of I_{Ln} at $t = 0^+$ is 1 normalised unit of current per unit of normalised time. Therefore the normalised current would have reached 1 unit at $t_n = 1$ if the rate of rise had remained unchanged. One unit of normalised time amounts to one τ of real time. Thus time constant is the time the current in an initially relaxed *RL* circuit would have taken to reach the final steady value had the initial rate of rise been maintained throughout. Equivalently, it is the time the voltage across inductor would have taken to reach zero if the initial rate of fall could be maintained throughout.

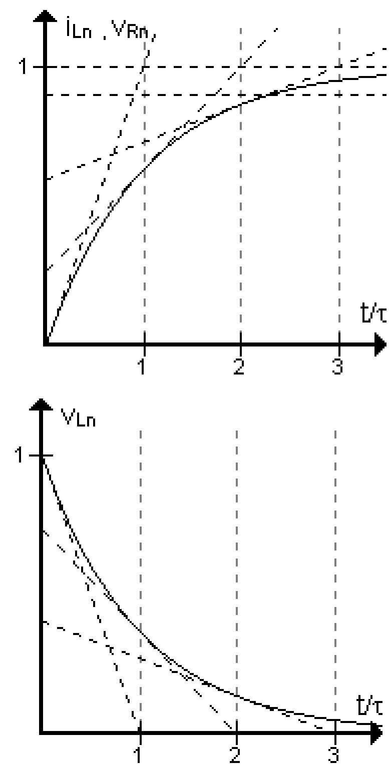

Fig. 10.5-2 Current Slope Based Interpretation of Time Constant

However, time constant is even more than that. Consider the remaining two tangents at $t_n = 2$ and $t_n = 3$ in Fig. 10.5-2. Moving along either tangent line will take us to the final value of inductor current in one unit of normalised time away from the instant at which the tangent is drawn. If this is true about these two time instants, then, it must be true for all time instants since there is nothing special about these two. Let us examine the slope of inductor current in detail.

$$i_{Ln}(t) = (1 - e^{-t_n})\ ;\ t_n \geq 0^+$$

$$\therefore \frac{di_{Ln}(t)}{dt_n} = e^{-t_n}\ ;\ t_n \geq 0^+$$

Let Δt_n be the time required from t_n to reach the final value of 1 with rate of change of $i_{Ln}(t)$ held at this value. Then,

$$\Delta t_n = \frac{1-(1-e^{-t_n})}{e^{-t_n}} = 1$$

$$\therefore \Delta t = \tau \text{ sec}$$

Thus, time constant of RL circuit is the additional time required from the current instant for the step response in the initially relaxed circuit to reach the final steady value assuming that the rate of rise of response is held constant at its current value from that instant onwards.

Rise Time and Fall Time in First Order Circuits

'Rise time (t_r)' and 'Fall time (t_f)' are two measures of time delay defined in the context of unit step response for linear dynamic systems. These two measures are quite general in definition in order to accommodate wide variety of systems having many terms in their transient response. However, in the case of simple first order systems there is a direct relationship between the time constant and rise and fall times.

Rise time is defined as the time interval between the first 10% point and first 90% point in the rising step response of a system where the percentages are to the base of the final step response value. Similarly, fall time is defined as the time interval between the first 90% point and the first 10% point in the step response of a system where the response variable is such that it starts at a non-zero initial value and decays to zero value in the long run. The percentages are to the base of the initial response value in this case.

These two definitions are illustrated in the case of series *RL* circuit step response in Fig. 10.5-3. Normalised variables are used in this figure and the corresponding normalised time points at which the 10% and 90% crossover takes place are also marked in the figure. From the figure it is clear that rise time and fall time of this circuit are equal to $\approx 2.2\tau$ sec, where τ is the time constant of the circuit. This result is valid for any circuit described by a first order linear differential equation with constant coefficients and has nothing to do with the inductive nature of the circuit under consideration.

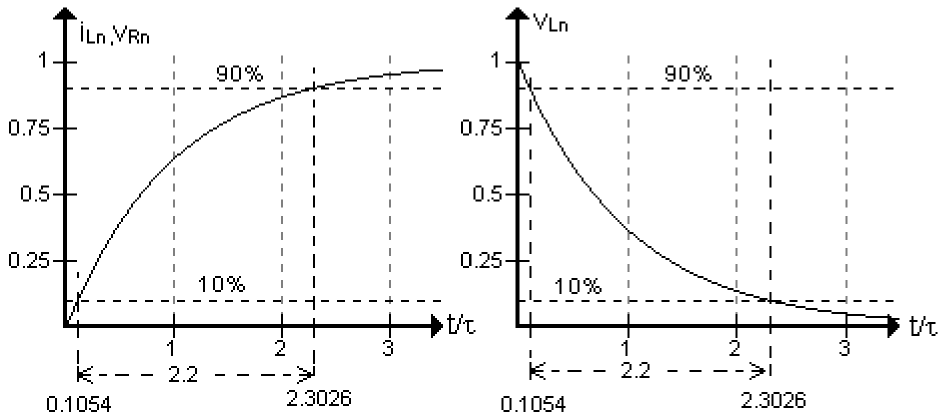

Fig. 10.5-3 Rise Time and Fall Time in Series *RL* Circuit

Rise time and fall time of the Series *RL* circuit are equal to $\approx 2.2\tau$ sec, where $\tau = L/R$ is the time constant of the circuit.

These two measures are defined for a general circuit of any order and therefore serve as measures of delay in response and depth of memory in the circuit in situations where a single time constant can not be identified as the major delaying factor in the circuit.

Effect of Non-Zero Initial Condition on Step Response of *RL* Circuit

We have been dealing with the step response of initially relaxed *RL* circuit till now. We have carefully included this constraint in our interpretation of time constant, definition of rise and fall times etc. We will generalise our understanding in this sub-section by bringing in non-zero initial current in the inductor at $t = 0^-$.

We have already derived the expression for inductor current in this case as

$$i_L(t) = I_0\, e^{-t/\tau} + \frac{1}{R}(1 - e^{-t/\tau}) \ \text{ for } t \ge 0^+ \qquad (10.5\text{-}4)$$

Normalising this equation by using *1/R* amps as the current base and τ sec as the time base we get the normalised form of the equation with I_{0n} as the normalised initial current at $t = 0^-$.

$$\begin{aligned} &i_{Ln}(t) = I_{0n}\, e^{-t_n} + (1 - e^{-t_n}) \ \text{ for } t_n \ge 0^+ \\ &\text{where } i_{Ln}(t) = \frac{i_L(t)}{1/R},\ I_{0n} = \frac{I_0}{1/R} \ \text{ and } t_n = \frac{t}{\tau} \end{aligned} \qquad (10.5\text{-}5)$$

This expression can also be written in a form that shows the natural response (transient response) and forced response components clearly separated out as below.

$$i_{Ln}(t) = \underbrace{1}_{\text{forced response}} - \underbrace{(1 - I_{0n})\, e^{-t_n}}_{\text{natural response}} \ ;\ \text{for } t_n \ge 0^+ \qquad (10.5\text{-}6)$$

Normalised step response current in Series *RL* circuit with the forced response and natural response components identified clearly

This equation is plotted in Fig. 10.5-4 with solid curves showing the total response and dotted curves showing the natural response or transient response. Curves are shown for four values of initial condition at $t = 0^-$. They are –0.5, 0, 0.5 and 1.5. All values are normalised ones. A negative initial condition value indicates that the initial current in the inductor at $t = 0^-$ was in a direction opposite to that of forced response. The forced response in all cases is represented by a horizontal line with intercept of unity in the vertical axis.

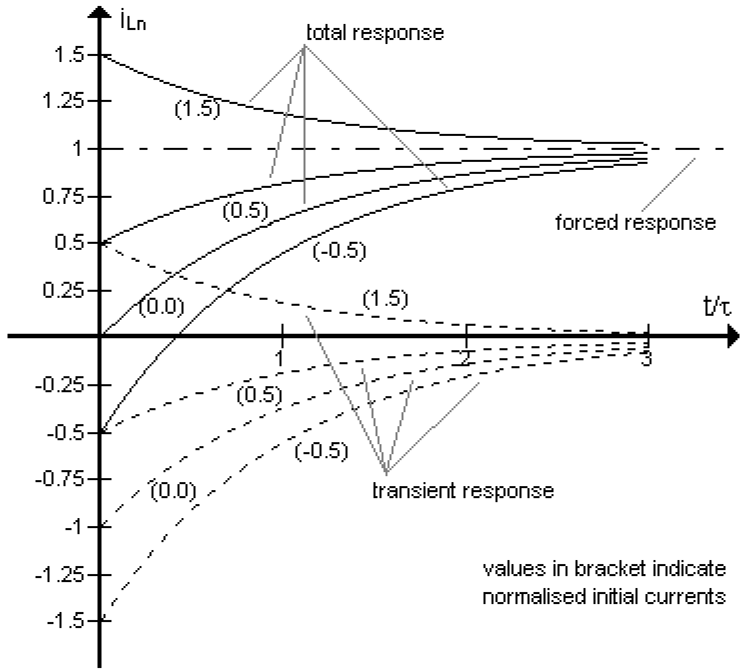

Fig. 10.5-4 Total Response and Transient Response in Series *RL* Circuit Step Response for Various Initial Currents

The role of transient response is seen to be one of bridging the gap between the initial current in the inductor and the final current in the inductor.

The waveforms in Fig. 10.5-4 and Eqn. 10.5-6 bring out the following aspects of *RL* circuit step response with non-zero initial condition.

(a) The transient response (natural response) of *RL* circuit contains two contributions – one from the initial condition specification and another from the value of forced response at $t = 0^+$. The magnitude of the transient term is decided by these two quantities. Transient response thus enforces compliance with the initial condition specification in the circuit.

(b) The total response is a rising response if the initial current at $t = 0^-$ is less than the final current value. It is a falling response if initial current is more than final current.

(c) There will be no transient response in the circuit if the initial current specified at $t = 0^-$ is equal to the final current value in magnitude and direction.

(d) Consider a new current variable defined as $\Delta i_{Ln}(t) = i_{Ln}(t) - I_{0n}$, *i.e.*, the change in inductor current from its initial value. Substituting Eqn. 10.5-6 in this definition we get

$$\Delta i_{Ln}(t) = (1 - I_{0n})(1 - e^{-t_n})\ ; \quad \text{for } t_n \geq 0^+ .$$

(e) Compare this expression for the change in inductor current with the inductor current expression for initially relaxed circuit. We can see that whatever that has been said about time constant becomes applicable in relation to the change in inductor current rather than to the total current when initial current is non-zero. The final value of this change is $1\text{-}I_{0n}$ and the change in inductor current rises to 63.2% of its final value in one time constant, 86.5% of its final value in 2τ sec etc. Similarly, the change in inductor current covers the 10% to 90% range in 2.2τ sec where the percentages are to the base of $1\text{-}I_{0n}$.

Free Response of Series *RL* Circuit

We consider a special case of an *RL* circuit with zero forcing function in this sub-section. Obviously, the solution for inductor current in this *source-free RL* circuit will contain only complementary solution. The particular integral is zero since forcing function is zero. The complementary solution is of the form $Ae^{-\alpha t}$ where α = *R/L*. Applying initial condition to this solution makes it clear that *A* =0 unless the initial condition specified at $t = 0^-$ is non-zero. Thus a source-free *RL* circuit can have a non-zero solution only if the inductor has some energy trapped in it at $t = 0^-$. This energy storage must have been created by some source prior to $t = 0^-$.

Consider the circuit (a) in Fig. 10.5-5. The switch S_1 was closed long back and the circuit has attained the final inductor current value of *1/R* amps by the time $t = 0^-$ is reached. At t = 0 the switch S_1 is opened and the switch S_2 is closed simultaneously. Thus a source-free series *RL* circuit with an initial current of I_0 (which is equal to *1/R* in the circuit in Fig. 10.5-5) is set up at *t* = 0. The circuit (b) is equivalent to circuit (a) for $t = 0^+$.

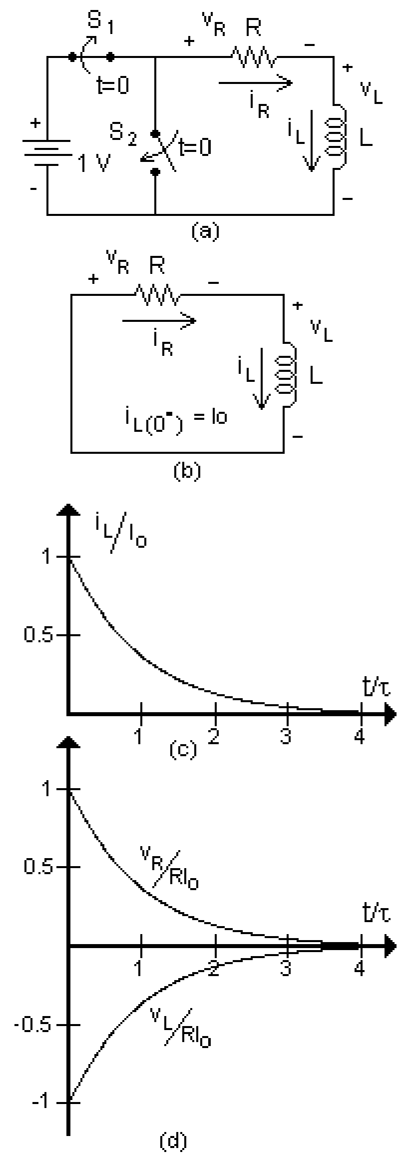

Fig. 10.5-5 Source-free *RL* Circuit and Waveforms

The expressions for inductor current and circuit voltages are derived as below.

$i_L(t) \quad = Ae^{-t/\tau}$; for $t \geq 0^+$

Since infinitely large voltage is neither applied nor supported in the circuit,

$$i_L(0^+) = i_L(0^-)$$

$$\therefore i_L(0^+) = I_0$$

$$\therefore A = I_0$$

$$\therefore i_L(t) = I_0 e^{-t/\tau} \text{ ; for } t \geq 0^+$$

$$i_R(t) = i_L(t) \ (\because \text{R and L are in series})$$

$$v_R(t) = RI_0 e^{-t/\tau} \text{ ; for } t \geq 0^+$$

$$v_L(t) = -v_R(t) \ \text{(By KVL)}$$

The current in the circuit decays exponentially from I_0 to zero with a time constant equal to *L/R* seconds. This is shown in (c) of Fig. 10.5-5. The corresponding voltage across inductor is negative valued and decays with the same time constant. The circuit voltages are shown in (d) of Fig. 10.5-5.

Example : 10.5-1

Obtain an expression for voltage across the resistor in an initially relaxed series *RL* circuit for rectangular pulse voltage input defined as $v_S(t)$ = 1 V for $0 \leq t \leq T$ and 0 V elsewhere. Use Integrating Factor Method. Plot the response for (i) $T = 0.2\tau$ (ii) $T = \tau$ and (iii) $T = 2\tau$.

Solution

The differential equation describing the circuit is

$$\frac{di_L}{dt} + \alpha i_L = \beta v_s(t) \text{ where } \alpha = \frac{1}{\tau} = \frac{R}{L} \text{ and } \hat{a} = \frac{1}{L}$$

or

$$di_L + \alpha i_L \, dt = \beta v_s(t) dt$$

The integrating factor for this equation is $e^{\alpha t}$. Multiplying the above equation by the integrating factor on both sides

$$e^{\alpha t} di_L + \alpha i_L e^{\alpha t} dt = \beta e^{\alpha t} v_s(t) dt$$

We identify the left side of the above equation as the exact differential of $i_L e^{\alpha t}$ since $d(i_L e^{\alpha t}) = e^{\alpha t} di_L + \alpha i_L e^{\alpha t} dt$.

Therefore

$$d(i_L e^{\alpha t}) = \beta v_s(t) e^{\alpha t} dt$$

Integrate the above equation between 0^+ and *t* to yield

$$(i_L e^{\alpha t})_{(t)} - (i_L e^{\alpha t})_{(0^+)} = \beta \int_{0^+}^{t} v_s(t).e^{\alpha t} dt$$

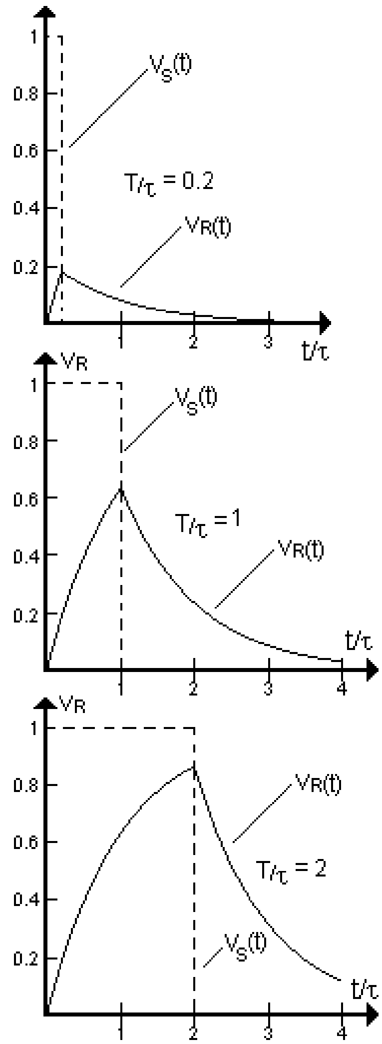

Fig. 10.5-6 Single Pulse Response of *RL* Circuit in Example : 10.5-1

Since the circuit is initially relaxed, the second term on the left side is zero. The input function is defined in a piece-wise manner and requires two expressions in two time ranges to define it. Therefore the integral on the right side has to evaluated separately for the two intervals $[0^+, T^-]$ and $[T^+, \infty)$.

$$\therefore\ i_L e^{\alpha t} = \beta \int_{0^+}^{t} 1.e^{\alpha t}\,dt \ \text{ for } 0^+ \le t \le T^-$$

$$\therefore\ i_L(t) = \frac{\beta}{\alpha}(1-e^{-\alpha t}) = \frac{1}{R}(1-e^{-t/\tau}) \ \text{ for } 0^+ \le t \le T^-$$

and ,for $t \ge T^+$,

$$i_L e^{\alpha t} = \beta\left[\int_{0^+}^{T^-} 1.e^{\alpha t}\,dt + \int_{T^-}^{T^+}(a\ \ bounded\ \ quantity).e^{\alpha t}\,dt + \int_{T^+}^{t} 0.e^{\alpha t}\,dt\right] \text{ for } T^+ \le t < \infty$$

$$\therefore\ i_L e^{\alpha t} = \frac{\beta}{\alpha}(e^{\alpha T}-1) \ \text{ for } T^+ \le t < \infty$$

$$= \frac{\beta}{\alpha} e^{\alpha t}\,(e^{-\alpha(t-T)} - e^{-\alpha t}) \ \text{ for } T^+ \le t < \infty$$

$$\therefore i_L(t) = \frac{\beta}{\alpha}(e^{-\alpha(t-T)} - e^{-\alpha T} e^{-\alpha(t-T)}) \ \text{ for } T^+ \le t < \infty$$

$$= \frac{1}{R}(1-e^{-\alpha T})\, e^{-\alpha(t-T)} \ \text{ for } T^+ \le t < \infty$$

Therefore the expression for $v_R(t)$ is

$$v_R(t) = \begin{cases}(1-e^{-\alpha t}) \text{ for } 0^+ \le t \le T^- \\ (1-e^{-\alpha T})\, e^{-\alpha(t-T)} \ \text{ for } T^+ \le t < \infty\end{cases}$$

$$= \begin{cases}(1-e^{-t/\tau}) \text{ for } 0^+ \le t \le T^- \\ (1-e^{-T/\tau})\, e^{-(t-T)/\tau} \ \text{ for } T^+ \le t < \infty\end{cases}$$

$$= \begin{cases}(1-e^{-t_n}) \text{ for } 0^+ \le t \le T^- \\ (1-e^{-T_n})\, e^{-(t_n-T_n)} \ \text{ for } T^+ \le t < \infty\end{cases}$$

The subscript '*n*' indicates normalisation with respect to τ. The plots of resistor voltage with normalised time for various T/τ ratios are shown in Fig. 10.5-6.

Example : 10.5-2

Repeat the problem in Example : 10.5-1 by solving for particular integral and complementary function.

Solution

The differential equation for $i_L(t)$ for the interval $[0^+, T^-]$ is

$$\frac{di_L}{dt} + \alpha\, i_L = \beta \ \text{ where } \alpha = \frac{1}{\tau} = \frac{R}{L} \text{ and } \hat{a} = \frac{1}{L}$$

Particular integral is the solution obtained by assuming that the forcing function was applied at infinite past. Hence particular integral value is $\beta/\alpha = 1/R$. The complementary function is of the form $A\,e^{-t/\tau}$. The circuit is initially relaxed. Applying initial condition to total solution and solving for A, we get the total solution as

$$i_L(t) = \frac{1}{R}(1-e^{-t/\tau}) \ \text{ for } \ 0^+ \le t \le T^- \qquad (10.5\text{-}7)$$

The inductor current would have followed this expression till there is a change in input source function or circuit structure. There is a change in the applied voltage at $t = T$ in the present example. The voltage applied for all $t \ge T^+$ is zero. Thus, the circuit is described by the following differential equation for $[T^+ \le t < \infty)$.

$$\frac{di_L}{dt} + \alpha\, i_L = 0 \;\text{ where } \alpha = \frac{1}{\tau} = \frac{R}{L} \;\text{ and } \hat{a} = \frac{1}{L}$$

The particular integral (*i.e.,* the solution if 0 is applied from infinite past) for this equation is zero. The complementary function is again $A\,e^{-(t-T)/\tau}$ (but valid only for $t \geq T^+$) with the value of A to be decided. The value of A is found out from the value of current at $t = T^+$. But since there was no impulse voltage involved in the circuit at $t = T$, the value of current at $t = T^+$ and $t = T^-$ will be same. This value can be obtained by substituting $t = T$ in Eqn. 10.5-7.

$$\therefore \text{Initial Condition for Current at } t = T^+ \;= \frac{1}{R}(1 - e^{-T/\tau})$$
$$\therefore\; i_L(t) = \frac{1}{R}(1 - e^{-T/\tau})e^{-(t-T)/\tau} \;\text{ for }\; T^+ \leq t < \infty \qquad (10.5\text{-}8)$$

We can get the expression for resistor voltage by multiplying the expressions for current by R. The solution is the same one we worked out in Example : 10.5-1 and is not repeated here.

Example : 10.5-3

Solve for *i* and *v* as functions of time in the circuit in Fig. 10.5-7.

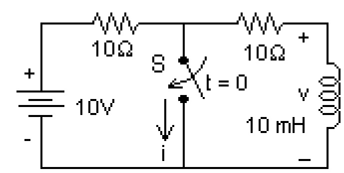

Fig. 10.5-7 Circuit for Example : 10.5-3

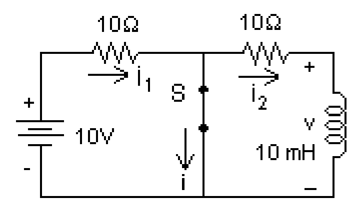

Fig. 10.5-8 Circuit for solving Example : 10.5-3

Solution

This circuit was already in a dc steady-state at t = 0. At t = 0 the switch closes, thereby forming a source-free *RL* circuit on the right side and a simple resistive circuit on the left side. These two circuits do not interact after t =0 except that the current through the switch will be a combination of the currents from these two circuits.

Inductor is a short for dc steady-state. Therefore, the initial current in the inductor at $t = 0^-$ was 10V/20Ω = 0.5A from top to bottom. Since the switching at t = 0 does not involve impulse voltage, the inductor current remains at 0.5A at $t = 0^+$ too.

Thus, a source-free *RL* circuit with initial current of 0.5A is set up at t = 0. The various current components in the circuit after t =0 are marked in Fig. 10.5-8.

$$i_1 = \frac{10\text{V}}{10\text{Ù}} = 1\text{A} \;;\; i_2 = 0.5e^{-1000t}\ \text{A}$$
$$\therefore \text{ The current through the switch } \; i = i_1 - i_2 = 1 - 0.5e^{-1000t} \;\text{ for }\; t \geq 0^+$$
$$v = 0.01\frac{di_2}{dt} = -5e^{-1000t}\ \text{V for }\; t \geq 0^+$$

Example : 10.5-4

Solve the circuit in (a) of Fig. 10.5-9 for the current through the switch as a function of time.

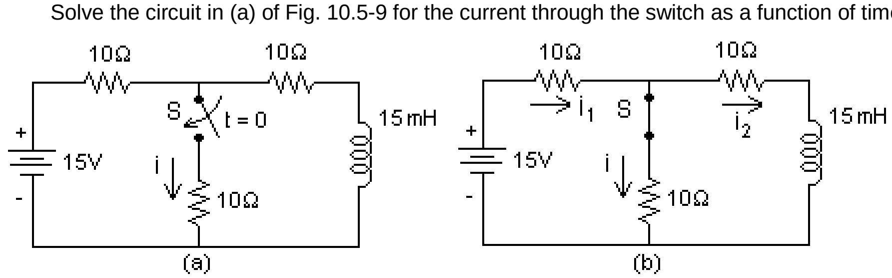

Fig. 10.5-9 Circuits for Example : 10.5-4

Solution

In this example the two meshes in the circuit interact after t = 0. We can solve this circuit in many ways – branch-current method, mesh analysis, Thevenin's equivalent etc. are some possibilities. First we solve it by branch-current method.

Various branch currents in the circuit are identified in (b) of eα t Fig. 10.5-9. We have to get a differential equation in the current variable i.

Applying KCL at switch node gives us $i_2 = i_1 - i$.

Applying KVL in the first mesh gives us $15 = 10(i_1 + i) \Rightarrow i_1 = 1.5 - i$

$\therefore\ i_2 = 1.5 - i - i = 1.5 - 2i$

Applying KVL in the second mesh gives us $10\,i = 10\,i_2 + 0.015\,(di_2/dt) = 15\text{-}20i - 0.03(di/dt)$

Therefore the differential equation governing i is $di/dt + 1000\,i = 500$ for $t \geq 0^+$.

Initial condition for i, *i.e.*, its value at $t = 0^+$ is needed. Initial value of i_2 is 15V/20Ω = 0.75A since the circuit was in dc steady-state prior to switching. Since $i_2 = 1.5 - 2i$, value of i at $t = 0^+$ will be (1.5 – 0.75)/2 = 0.375A. The particular integral of the differential equation for i is 500/1000 = 0.5A. Time constant is 1/1000.

Therefore $i = C\,e^{-1000\,t} + 0.5$. Evaluating C from initial condition for i at $t = 0^+$, we get $C = 0.375 - 0.5 = -0.125$.

Therefore the switch current $i = 0.5 - 0.125\,e^{-1000\,t}$ for $t \geq 0^+$.

Let us solve the same problem by mesh analysis. The relevant circuit with two mesh currents – I_1 and I_2 – identified is shown in (a) of Fig. 10.5-10 .

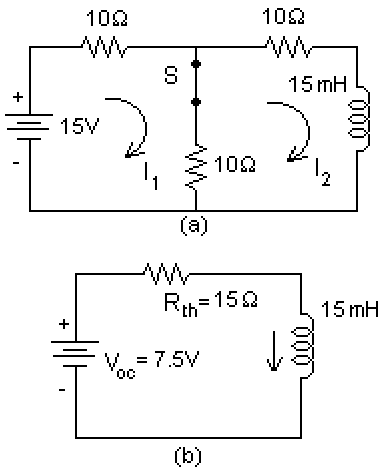

Fig. 10.5-10 Circuits for Mesh Analysis and Thevenin's Equivalent Analysis in Example : 10.5-4

The two mesh equations are

$$20I_1 - 10I_2 = 15$$

$$20I_2 - 10I_1 + 0.015\frac{dI_2}{dt} = 0$$

Eliminating I_1 from second equation using the first equation and simplifying, we get

$$\frac{dI_2}{dt} + 1000\,I_2 = 500$$

The initial condition for I_2 at $t = 0^+$ is same as the initial condition for inductor current at that instant. This value is 0.75A.Particular integral is 0.5A. Time constant is 1/1000 sec.

$$I_2 = C\,e^{-1000t} + 0.5$$
$$C = 0.25 \text{ and } I_2 = 0.5 + 0.25e^{-1000t}$$
Using this solution in the first mesh equation we can get
$$I_1 = 1 + 0.125e^{-1000t}$$
$$\therefore \text{Current through the switch} = I_1 - I_2 = 0.5 - 0.125e^{-1000t} \text{ A for } t \geq 0^+$$

This circuit problem can be solved by using Thevenin's theorem too.

The circuit portion to the left of inductor may be replaced by its Thevenin's equivalent as shown in (b) of Fig. 10.5-10. Inductor current can be obtained from this circuit. Once inductor current is available we will be able to get back to the switch current using KCL or KVL. This is illustrated now.

The initial current in the inductor is 0.75A again. The circuit in (b) of Fig. 10.5-10 is a simple series *RL* circuit and its particular integral is 7.5/15 = 0.5A. Its time constant is 15mH/15Ω = 1ms. Therefore its solution is = $C\,e^{-1000\,t} + 0.5$. Evaluating the initial condition constant C and completing the solution, we get, inductor current = $0.5 + 0.25\,e^{-1000\,t}$A.

$$i_L = 0.25e^{-1000t} + 0.5 \text{ A}$$
$$\text{Voltage across 10Ù in the switch path} = 10\times(0.25e^{-1000t} + 0.5) + 0.015\times(-250e^{-1000t})$$
$$= 5 - 1.25e^{-1000t} \text{ V}$$
$$\therefore \text{Current through the switch} = (5 - 1.25e^{-1000t}) \text{ volts/10Ù}$$
$$= 0.5 - 0.125e^{-1000t} \text{ A for } t \geq 0^+$$

We derived the differential equations governing three variables in the circuit – the branch current in the central limb, second mesh current and the inductor current in the process of solving this circuit. The left-hand side of all the three differential equations had the same coefficients. (Why ?)

We also notice that the time constant of the circuit can be easily found as L/R_{th} where R_{th} is the Thevenin's equivalent resistance appearing across the inductor. But Thevenin's equivalent is found by deactivating all independent sources.

Therefore, the time constant of a single-inductor circuit can be found by replacing all independent voltage sources by short-circuits and all independent current sources by open-circuits and finding the equivalent resistance connected across the inductor. We illustrate this procedure further in the next example.

Example : 10.5-5

Show that the current in 16mH inductor in the circuit (a) in Fig. 10.5-11 will go to zero as $t \to \infty$. Also find the inductor current and currents delivered by the voltage sources as functions of time. Find out how long we have to wait for the inductor current to fall below 100mA.

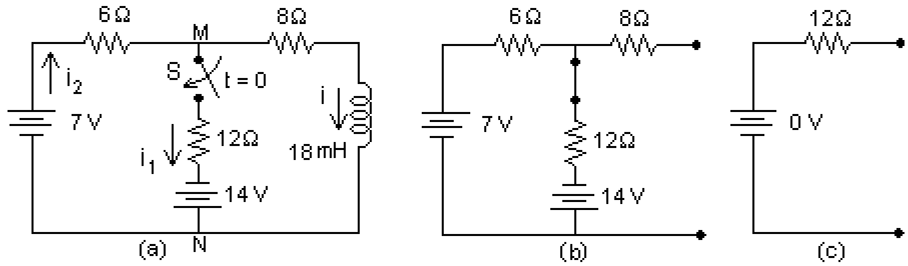

Fig. 10.5-11 Circuits for Example : 10.5-5 (a) Circuit for the Problem (b) Circuit for Finding Thevenin's Equivalent (c) Thevenin's Equivalent

Solution

First we find the time constant effective after $t = 0^+$. The Thevenin's equivalent of the circuit connected across the inductor is evaluated by using the circuit in (b) and the resulting equivalent is shown in (c) of Fig. 10.5-11. Since the voltage source in Thevenin's equivalent is zero-valued the inductor current will have a zero steady-state value. The time constant of the circuit is 16mH/12Ω = 1.5ms.

We find the initial condition for inductor current next. The circuit was in steady-state prior to switching at $t = 0$. Inductor is replaced by a short-circuit for dc steady-state. Therefore the inductor current at $t = 0^-$ must have been 7/14 = 0.5A and it will be 0.5A at $t = 0^+$ since there is no impulse voltage involved in the switching.

Now the circuit is a series *RL* circuit with a known initial condition and dc sources. We know the solution for such a circuit. It is of general format – $A\,e^{-t/\tau}$ + C - where *C* is the particular integral (therefore, the dc steady-state value) and *A* is the arbitrary constant to be found out from initial condition. *This is the general format of solution for any circuit variable in a first order circuit with dc excitation.*

$$\therefore i(t) = Ae^{-t/1.5} + 0;\ t \text{ in ms}$$
$$i(t) = 0.5 \text{ at } t = 0^+$$
$$\therefore i(t) = 0.5e^{-t/1.5}\ \text{A};\ t \text{ in ms and } t \geq 0^+ \qquad (10.5\text{-}9)$$
$$\therefore \text{Voltage across inductor} = 0.018\frac{di}{dt} = -6e^{-t/1.5}\ \text{V};\ t \text{ in ms and } t \geq 0^+$$

We have two ways to find the currents delivered by the voltage sources - i_1 and i_2. In the first method we find the voltage across M and N in circuit (a) as $v_{MN} = 8i + 0.018\,di/dt$ and then find i_2 as $(7 - v_{MN})/6$ and i_1 as $(v_{MN} - 14)/12$.

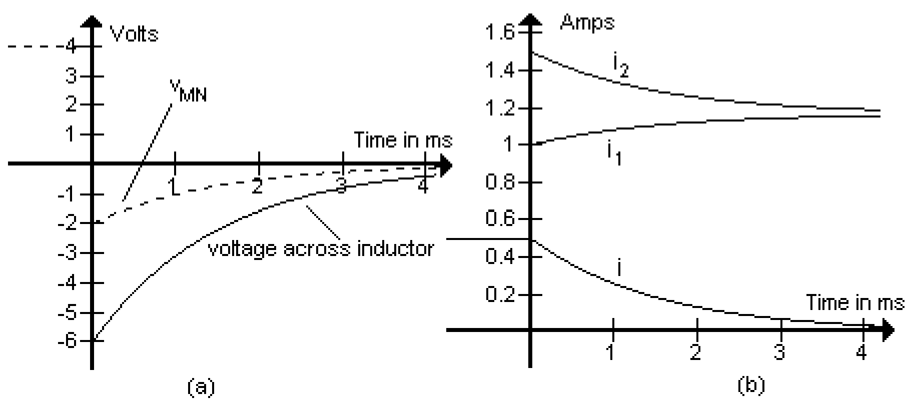

Fig. 10.5-12 Waveforms of (a) voltages and (b) source and inductor currents in Example : 10.5-5

In the second method, we realise that all variables in this circuit will have a $A\,e^{-t/\tau}$ term and a steady-state term and that we can find the arbitrary constant *A* if we know the value of the particular variable at 0^+. So we set out to find the initial and final (*i.e.*, steady-state) value of the source currents. The inductor current was at 0.5A at $t = 0^+$. The voltage across inductor at that instant is –6V from Eqn. 10.5-9. Therefore v_{MN} = 8x0.5 - 6 = -2 V at $t = 0^+$. Therefore initial value of i_1 at $t = 0^+$ = (-2 – (-14))/12 = 1A. Final current in inductor is zero. Hence final current in both sources will be same and equal to (7+14)V/(6+12)Ω = 7/6 = 1.167A.

Therefore we required currents are

$$\begin{aligned} \therefore i_1(t) &= Ae^{-t/1.5} + 1.167;\ t \text{ in ms with } i_1 = 1\text{amp at } t = 0^+ \\ \therefore i_1(t) &= 1.167 - 0.167e^{-t/1.5} \text{ amp};\ t \text{ in ms and } t \geq 0^+ \\ i_2(t) &= i_1(t) + i(t) = 1.167 - 0.167e^{-t/1.5} + 0.5e^{-t/1.5} \\ &= 1.167 + 0.333e^{-t/1.5} \text{ amp};\ t \text{ in ms and } t \geq 0^+ \end{aligned}$$

These are plotted in Fig. 10.5-12. The time required for inductor current to go below 100mA is found as follows.

$$0.1 = 0.5e^{-t/1.5} \text{ A};\ t \text{ in ms},\ \therefore -\frac{t}{1.5} = \ln 0.2 = -1.6094 \text{ and } t = 2.414 \text{ ms}$$

10.6 Steady-State Response and Forced Response

The total response in *RL* circuit to any forcing function will consist of two terms – the transient response (or natural response) and the forced response. The transition from initial state of the circuit (which is encoded in a single number in the form of initial inductor current specification at $t = 0^-$) to the final state (in which only forced response will be present) is accomplished with the help of the transient response. Now we introduce a new term called steady-state response and relate it to the response terms we are already familiar with.

Our study of the solution of differential equation describing the *RL* circuit has shown us that the total response will always contain two components – the transient response and the forced response. Of course, forced response will be zero if forcing function is zero *i.e.*, in a source-free circuit. Similarly, the transient response term may become zero under certain suitable initial condition values. But these are special situations and, in general, there will be two terms in the total response. This is true not only for *RL* circuit but also for any linear circuit described by linear ordinary differential equations with constant coefficients. Such a circuit of higher order will have two groups of terms in its total solution – first group constituting transient response containing one or more terms and the second group constituting forced response containing one or more terms depending on the type of forcing function. *Thus forced response is a response component which is always present in the total response of a circuit except when the forcing function itself is zero.*

> **Steady-State**
>
> **A circuit is said to have reached steady-state with respect a particular forcing function if all transient response terms decay down to negligible level and the only response component that remains is the forced response component.**

We have seen that the transient response of *RL* circuit contains exponential function of the form $e^{-\alpha t}$ where α is a +ve number decided by *R* and *L*. Such an exponential function with negative real index will taper down towards zero as *t* approaches ∞. Hence we expect the transient response in an *RL* circuit to vanish with time quite irrespective of the forced response component. Therefore, we expect that there will only be the forced response component active in the circuit in the long run, *i.e.*, after sufficient time had been allowed for the transient response to die down. When all the transient response terms in all the circuit variables in the circuit have died down to negligible levels (they never die down to zero) and the only response component in all the circuit variables is the forced response component, we say the circuit has reached the steady-state with respect to the particular forcing function that was applied to the circuit. Notice that under *steady-state* conditions the transient response terms should not be present in any circuit variable at all. Or, in other words, there can not be a circuit which attains steady-state in some of its variables and does not attain steady-state in yet others.

Therefore, a circuit will reach steady-state if and only if all its transient response terms are of decreasing type. Moreover, the only response that will continue in the circuit after it has reached steady-state is the forced response component. Therefore, *steady-state response* is same as *forced response* with the condition that the steady-state will exist only if all the transient response terms are of damped nature - *i.e.*, decreasing functions of time. Thus, steady-state response is another name for forced response when transient response is assured to die down to negligible levels. Forced response will always be present; but steady-state response need not be.

Consider the circuit in Fig. 10.6-1. The 1V source on the left side has set up an initial current of 1A in the inductor of 1H at $t = 0^-$. The switch S_1 is opened and switch S_2 is closed at $t = 0$ to apply a 1V source right across the inductor. The current in the inductor is shown in (b) of Fig. 10.6-1. We note that with a bounded input (1V d.c source is a bounded input) we get a current in the inductor which is not bounded. Also, the transient response ($1e^{-0\,t}$) does not decrease with time. Therefore there is no steady-state in this circuit though there is a forced response.

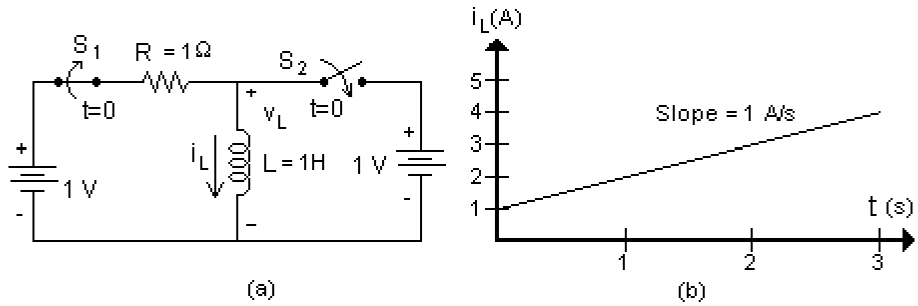

Fig. 10.6-1 A Circuit With No Steady-State and Its Step Response

The DC Steady-State

It is to be noted is that the steady-state attained by a circuit is intimately connected with the type of forcing function applied to the circuit. We can not expect the generalisations arrived at based on the steady-state behaviour for a particular forcing function to hold in the case of steady-state behaviour for another forcing function.

For example, consider the steady-state in series *RL* circuit when input is a unit step voltage. The forced response in this case is a constant current of value 1/*R* amps. The voltage across inductor with a constant current through it can only be zero. But zero is indeed a constant. Thus, we see that, under steady-state condition with step input, all the circuit variables become constants. The only constant voltage an inductor can have across it is zero if the current through it is also constrained to remain constant since $v = L$ di/dt for an inductor. Noticing the 'constant' nature of input voltage and 'constant' nature of all circuit variables under steady-state, we name this kind of steady-state as the *dc steady-state*.

Under dc steady-state (provided the circuit can reach such a steady-state - transient response has to die down for that) inductors in a circuit can have any constant-valued current; but their voltages will be constrained to remain at zero. But that is similar to the definition of a short-circuit – except that the voltage across a short-circuit is zero for *any* current. Thus we can solve for dc steady-state response in *RL* circuits (containing one or more inductors) by replacing all inductors by short-circuits – provided all independent sources in the circuit are switched dc sources or step functions and a dc steady-state can exist in the circuit. Subject to the above conditions, we can state that '*an inductor is a short-circuit under dc steady-state*'.

An inductor in a circuit can be replaced by a short-circuit for analysing the circuit under dc steady-state condition provided the circuit is capable of a dc steady-state.

While we are on the topic of steady-state, we might as well look at the remaining two kinds of steady-state we will need in the analysis of dynamic circuits.

The Sinusoidal Steady-State

Sinusoidal steady-state refers to the steady-state that gets established in a circuit when all the independent sources in the circuit are sinusoids of same angular frequency. Like dc steady-state, this steady-state too can exist only if all the terms in transient response die down to negligible levels with time.

The word '*steady*' has the literal meaning of '*unchanging*'. This unfortunately gives an impression that *steady-state* is that state in a circuit in which all circuit variables are *unchanging* in time. This is an error that a beginner in Circuit Analysis has to guard against. *Steady-State* does not necessarily mean that circuit response is *unchanging* in time. It is so only in the case of d.c steady-state.

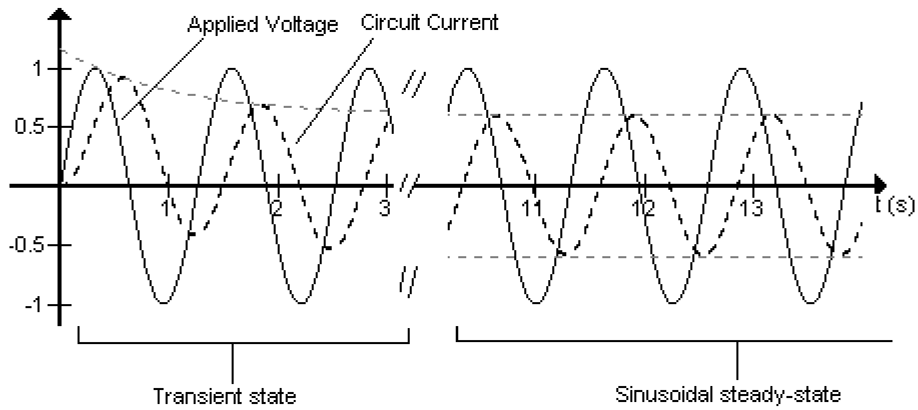

Fig. 10.6-2 Waveforms Illustrating Sinusoidal Steady-State

To understand the meaning of the word '*steady*' in the Circuit Analysis context, we have to look at the input forcing function and find out those features of the forcing function which remain unchanging with time. In the case of dc or step inputs, the input value itself is unchanging in $[0^+, \infty)$. In the case of sinusoidal input forcing function the amplitude of the sinusoid, its angular and cyclic frequencies, its phase and its shape in one period (*i.e.*, sinusoidal shape) remain constant in time. Therefore, if a steady-state exists in a circuit under the action of such a forcing function, we can expect all circuit variables to have sinusoidal shape, fixed amplitudes, fixed frequency *which is same as that of the forcing function* and fixed phase with respect to forcing function. This is what is meant by sinusoidal steady state.

Meaning of 'sinusoidal steady-state response'

Thus, a circuit excited by one or more sinusoidal forcing functions of *same* frequency is said to have reached *sinusoidal steady-state* if all its transient response components have died down and all its circuit variables have *sinusoidal* waveshape with *same* frequency as that of forcing functions and *fixed* amplitudes and phase angles.

The waveforms in Fig. 10.6-2 show the applied voltage and inductor current in an initially relaxed *RL* circuit with R = 0.33 Ω and L = 0.33 H. A sinusoidal voltage = 1 sin (*5t*) volts was switched on to the circuit at t =0. The current waveform shows the exponential transient response in the first few seconds clearly. After about 10 sec or so, the transient response has decayed to negligible level and the response contains only a sinusoidal waveform that is of same frequency as that of applied voltage. It has fixed amplitude and a fixed phase with respect to the input sine wave. Thus, the circuit has reached sinusoidal steady-state within few time constants (τ = 1sec).

Sinusoidal steady-state is also referred to as *ac steady-state* in Circuit Analysis Literature.

The Periodic Steady-State

This is the third kind of steady-state that can come up in linear circuits. Here the input forcing function is periodic but not sinusoidal. Therefore a single number like amplitude in the case of a sinusoid is not available for this input. The only aspect that is *steady* about it is its frequency. Thus we expect the circuit to reach a steady-state (if it can) in which all circuit variables will be *periodic* with same *frequency* as that of input. But the waveshape of response variables will not be the same as the waveshape of input forcing function. The sinusoidal steady-state we discussed in the previous sub-section is indeed a periodic steady-state; but it is more than that – in sinusoidal steady-state, the response waveform is same as that of forcing function. In a periodic steady-state such a constraint may not be satisfied.

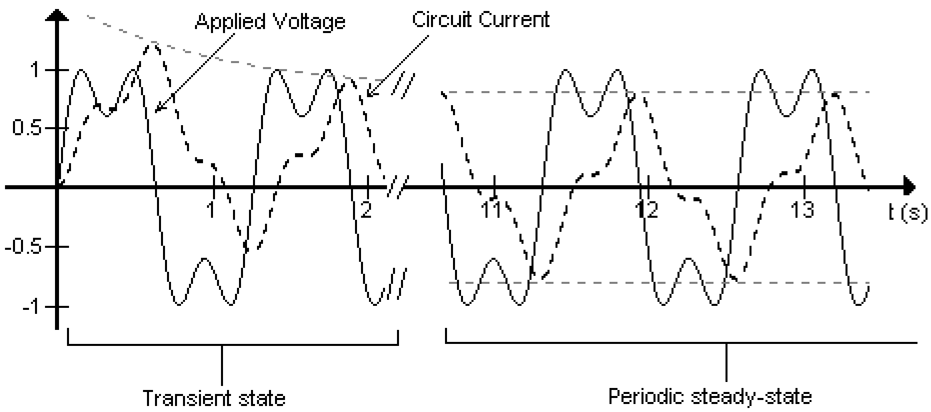

Fig. 10.6-3 Waveforms Illustrating Periodic Steady-State

The waveforms shown in Fig. 10.6-3 illustrate the attainment of periodic steady-state in an *RL* circuit (R = 0.33Ω, L = 0.33 H, $I_0 = 0$, τ = 1s) driven by a voltage = (1 sin 5t + 0.4 sin 15t) volts from $t = 0^+$ onwards. Notice that the waveshape of current is not the same as that of applied voltage. However the current is periodic with same period and its waveshape remains the same in successive periods after it has reached steady-state.

We saw that inductors can be replaced by short-circuits for dc steady-state analysis. However, we notice that the currents in the inductors in circuits are time-varying currents in sinusoidal steady-state and periodic steady-state conditions. *Hence inductors can not be replaced by short-circuits for all kinds of steady-state analysis – that works only for dc steady-state analysis.*

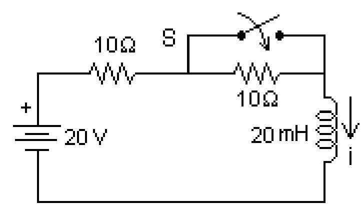

Fig. 10.6-4 Circuit for Example : 10.6-1

Example : 10.6-1

Periodic steady-state can come up in *RL* circuits when one or more sources in the circuit are periodic functions of time. But even when all sources are dc sources there can be a periodic steady-state in the circuit if some parameter in the circuit is varying periodically with time. We consider one such circuit in this example.

The circuit for this example appears in Fig. 10.6-4. The switch S in the circuit is operated periodically at a switching frequency of 1kHz. In every switching cycle it is kept closed for half the time and then kept open for remaining half of cycle time i.e., 0.5ms. After a large number of switching cycles a steady repeating pattern of current gets established in the circuit. We describe this steady pattern in this example.

Solution

Assume that the circuit is in periodic steady-state. Let the current start with a value I_1 when the switch is closed in the beginning of a steady-state cycle . Then the current increases along an exponential with τ of 2ms towards 2A (because 2 is the particular integral under this condition). But the current is not allowed to reach 2A since the switch goes open after 0.5ms when the current is I_2. Now a new transient starts, trying to take the current from this value to 1A (because 1 is the particular integral now) exponentially with a time constant of 1ms. But this exponential is terminated after another 0.5ms and then the cycle starts all over again. *At the end of the second 0.5ms, the current in the circuit will be I_1 if the circuit is in periodic steady-state.* We convert the above reasoning into equations and solve for I_1 and I_2.

$$I_1 e^{-0.5/2} + 2(1-e^{-0.5/2}) = I_2 \text{ and } I_2 e^{-0.5/1} + 1(1-e^{-0.5/1}) = I_1$$
$$\left[I_1 e^{-0.5/2} + 2(1-e^{-0.5/2})\right]e^{-0.5/1} + 1(1-e^{-0.5/1}) = I_1$$
$$\therefore I_1 = 1.254\,\text{A and } I_2 = 1.419\,\text{A}$$

The corresponding values for 500Hz switching also were worked out and they are I_1 = 1.186 A and I_2 = 1.506 A. The plot of inductor current for 1kHz switching and 500Hz switching are shown in (a) and (b) of Fig. 10.6-5.

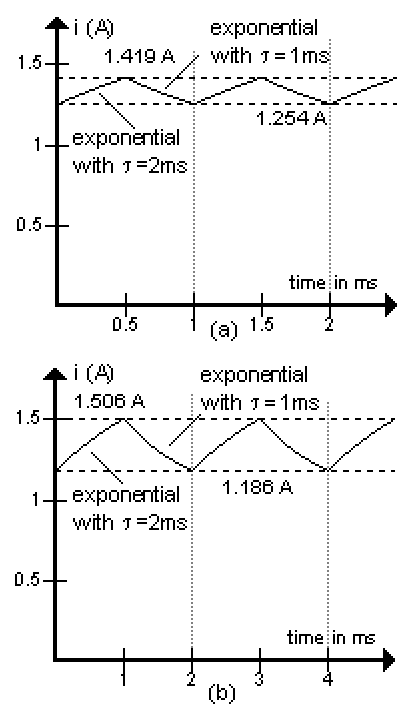

Fig. 10.6-5 Periodic steady-state in Circuit in Example : 10.6-1 for (a) 1kHz and (b) 500Hz switching

The total resistance in the circuit was changing abruptly between 10 and 20 ohms. But the current in the circuit does not show any discontinuity. This once again illustrates the fact that inductor smoothes a circuit current. Moreover, we see from Fig. 10.6-5 that, the smoothing effect is more when circuit time constants are larger than the switching cycle period. In fact, the current tends to become almost constant at 1.33A as inductance value in this circuit is increased. (The reader is encouraged to ponder over why it should it be 1.33A and why the average value of waveforms in Fig. 10.6-5 also should be 1.33A?)

10.7 Linearity and Superposition Principle in Dynamic Circuits

We have been dealing with *unit* step response of *RL* circuit till now. How do we get the solution if it is not *unit step*, but a step of size *V*? In short, how do we solve the series *RL* circuit if the input source function is $V\,u(t)$? Can we just multiply the unit step response by *V* to get the solution for this input?

We know that memory-less circuits containing linear passive resistors, linear dependent sources and independent sources will be linear and will obey superposition principle. We examine the issue of linearity of circuits containing one or more energy storage elements along with resistors, linear dependent sources and independent sources in this section. We are already familiar with a particular decomposition of total response in such a circuit in terms of transient response and forced response. We will see in this section that yet another decomposition of total response into the so-called *zero-input response* and *zero-state response* is needed in view of linearity considerations in the circuit.

An electrical element is called *linear* if its element equation obeys superposition principle. Superposition principle involves two sub-principles – principle of additivity and principle of homogeneity. We have seen earlier that an inductor with an element relation $v = L$ di/dt and a capacitor with an element relation $i = C$ dv/dt are linear elements. But will an interconnection of such linear elements (and independent sources) into a circuit result in a *linear system*? A linear system is one in which *every* response variable in the system obeys superposition principle.

Intuitively we expect an interconnection of linear elements and independent sources to yield a linear circuit; but mathematically it is not that simple. It requires to be proved. The proof involves slightly advanced concepts from a mathematical topic called *linear vector spaces* and we do not take up that here. *We accept the result that a circuit formed by interconnecting linear passive elements, linear dependent sources and independent sources will be a linear circuit. Such a linear circuit has to obey superposition principle.*

Therefore, we must be able to get $i_L(t)$ in a series *RL* circuit with $V\,u(t)$ volts as its input source function by scaling the unit step response by *V*. Assuming an initial condition of I_0 at $t = 0^-$, this scaling results in Eqn. 10.5-4 getting multiplied by a dimensionless scalar *V* to yield

$$i_L(t) = VI_0\, e^{-t/\tau} + \frac{V}{R}(1 - e^{-t/\tau}) \text{ for } t \geq 0^+$$

as the solution. But this solution is incorrect because the current at $t = 0^+$ is VI_0 according to this equation rather than the correct value of I_0 .It looks as if the principle of homogeneity is not valid here. Let us try to get the solution without resorting to linearity.

The complementary solution is $A\, e^{-\alpha t}$ with $\alpha = 1/\tau = R/L$ and the particular integral is V/R. Therefore the total solution is $i_L(t) = A\, e^{-\alpha t} + V/R$. Substituting the initial condition at $t = 0^+$ and solving for A, we get the final solution as $i_L(t) = (I_0 - V/R)\,e^{-\alpha t} + V/R = I_0\, e^{-\alpha t} + V/R\,(1 - e^{-\alpha t})$ for $t \geq 0^+$. Thus the correct solution is,

$$i_L(t) = I_0\, e^{-t/\tau} + \frac{V}{R}(1 - e^{-t/\tau}) \text{ for } t \geq 0^+ \qquad (10.7\text{-}1)$$

The first term in Eqn. 10.5-4 does not get multiplied by *V* when the step magnitude is scaled by *V*. The second term in the same equation gets scaled by *V*. This