Introduction

The last chapter showed that:

(1) All the element voltages and element currents in a circuit can be obtained from its node voltages. The node voltages are governed by a matrix equation ***YV*** = ***CU*** where ***V*** is the node voltage column vector, ***Y*** is the nodal conductance matrix of the circuit, ***U*** is the input column vector containing source functions of all *independent* voltage sources and current sources in the circuit and ***C*** is the input matrix. The values of conductances in the circuit and values of coefficients of linear dependent sources in the circuit decide the elements of ***Y***-matrix. It is a symmetric matrix if there are no dependent sources in the circuit. Dependent sources *can* make ***Y***-matrix asymmetric. The ***C*** matrix will, in general, contain 0, 1, -1 and conductance values as well as dependent source coefficients.

(2) An alternative formulation is given by a matrix equation ***ZI*** = ***DU*** where ***I*** is the mesh current column vector, ***Z*** is the mesh resistance matrix of the circuit, ***U*** is the input column vector containing source functions of all *independent* voltage sources and current sources in the circuit and ***D*** is the input matrix. All the element voltages and element currents in a circuit can be obtained from its mesh currents. The values of resistances in the circuit and values of coefficients of linear dependent sources in the circuit decide the elements of ***Z***-matrix. It is a symmetric matrix if there are no dependent sources in the circuit. Dependent sources *can* make ***Z***-matrix asymmetric. The ***D*** matrix will, in general, contain 0, 1, -1 and resistance values as well as dependent source coefficients.

(3) Any response variable in a circuit (*i.e.,* any element voltage or current or combination thereof) can be expressed as a *linear combination* of source functions of independent voltage sources and independent current sources present and active in the circuit. *i.e.,* $x = a_1I_1 + a_2I_2 + ... + b_1V_1 + b_2V_2 + ...$, where x is some chosen response variable and a's and b's are coefficients decided by circuit conductances/resistances, dependent source coefficients and interconnection details. Some of the a's and b's may turn out to be zero for certain choices of x.

These three basic properties of a circuit comprising linear elements are recast into various useful theorems that simplify the circuit analysis procedure in practical contexts. That is, the so-called circuit theorems are more or less restatements of these basic facts. These three observations were arrived at by analysis of memoryless circuits. However, we will show in later chapters that they are true for dynamic circuits too. Therefore, all important circuit theorems that we arrive at in this chapter too will be valid for dynamic circuits.

The crucial fact the reader should keep in mind is that we are stating nothing more in almost all circuit theorems (in many of them anyway) than what is already contained in the three properties described above.

Why Circuit Theorems?

Circuit Analysis involves determination of element voltages and currents in all elements of the circuit using element equations and interconnection equations. Kirchhoff's Current Law equations at all nodes and Kirchhoff's Voltage Law equations in all loops along with element *v-i* relationship equations will yield the necessary set of equations.

However, we need systematic procedures for exploiting these equations. Node analysis and Mesh analysis were two such systematic procedures we took up for detailed study in the last chapter. In this chapter, we discuss some circuit theorems and circuit transformations that increase our efficiency in solving circuits. Moreover, they render further insight into certain features of a *linear* circuit. These theorems constitute a basic set of tools that enhance the analyst's efficiency in solving circuits.

5.1 Linearity of a Circuit and Superposition Theorem

Consider a purely resistive circuit driven by two independent current sources and an independent voltage source shown in Fig. 5.1-1.

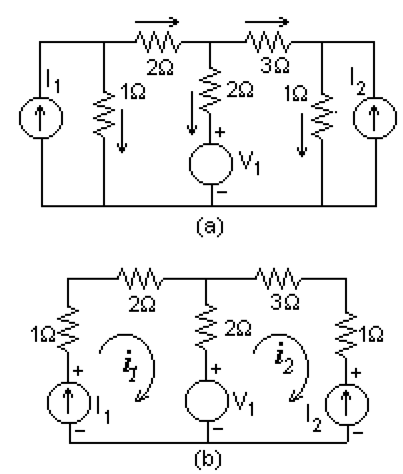

Fig. 5.1-1 (a) A Circuit with Three Independent Sources (b) Circuit after Source Transformation

The mesh equations for circuit (b) in matrix form can be derived as

$$\begin{bmatrix} 5 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}=\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ V_1 \end{bmatrix}$$

Solving the matrix equation, we get,

$$i_1 = \frac{3}{13}I_1 - \frac{1}{13}I_2 - \frac{2}{13}V_1$$

$$i_2 = \frac{1}{13}I_1 - \frac{5}{26}I_2 + \frac{3}{26}V_1$$

Now, any element voltage or current can be expressed in terms of these two mesh currents. For example, consider the current in resistor in the central limb in the direction shown in circuit (a) of Fig. 5.1-1. This current is obtained by applying KCL at the junction between the three resistors and is = i_1 - i_2 = $\frac{2}{13}I_1 + \frac{3}{26}I_2 - \frac{7}{26}V_1$.

Consider the current in the 1Ω across the current source I_1. This is obtained by applying KCL at the node where I_1, 2Ω and 1Ω are connected together. The current in 2Ω is same as the first mesh current in circuit (b) and hence the current in 1Ω = I_1-i_1 = $\frac{10}{13}I_1 + \frac{1}{13}I_2 + \frac{2}{13}V_1$.

Currents in all elements can be worked out in a similar manner. These currents are marked in Fig. 5.1-2 using a notation where the three numbers in brackets show the coefficients of I_1, I_2 and V_1 respectively. Or, they can be interpreted as the current components when all the three sources have *unit values.* Consider the numbers marked for the voltage source V_1. It is $(\frac{2}{13}, \frac{3}{26}, -\frac{7}{26})$. This means that I_1 contributes $\frac{2}{13}$ amps per unit amp to this current, I_2 contributes $\frac{3}{26}$ amps per unit amp to this current and V_1 contributes $-\frac{7}{26}$ amps per unit volt to this current.

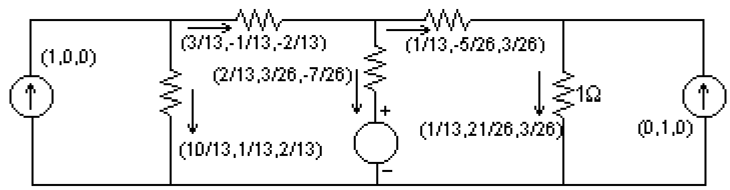

Fig. 5.1-2 Currents in Various Elements in the Circuit (a) of Fig. 5.1-1

Thus, each element current (and element voltage too) is made up of three components. Each independent source contributes one component to each element current and voltage. The components contributed by all the independent sources add together to form the total response.

That the total currents in elements and total voltage across them will satisfy KCL and KVL respectively is only to be expected. In fact, that was the basis for node analysis and mesh analysis. However, what is not so obvious is that *components contributed by a particular independent source to all element currents will satisfy KCL at all nodes without depending in any way on the components provided by other independent sources.* Similarly, *components contributed by a particular independent source to all element voltages will satisfy KVL in all loops without depending in any way on the components provided by other independent sources.* This may be verified easily in the example that was analysed in this section.

The contribution of a particular source to a circuit variable does not change when some other source/s is acting simultaneously along with it.

The implication from this observation is that the components contributed by a particular independent source to circuit variables are the same as the solution of the circuit when that source is acting alone without the other sources present. Or, in other words, the contribution of a particular source to a circuit variable does not change when some other source/s is acting simultaneously.

This can be understood in another way too. We had seen that all mesh current variables and node voltage variables (and hence all voltage variables current variables in the circuit) for a circuit can be expressed as linear combinations of independent source functions. The solution for mesh currents in the circuit in Fig. 5.1-1 was

$$\begin{aligned} i_1 &= \frac{3}{13}I_1 - \frac{1}{13}I_2 - \frac{2}{13}V_1 \\ i_2 &= \frac{1}{13}I_1 - \frac{5}{26}I_2 + \frac{3}{26}V_1 \end{aligned}.$$

Thus there are three contributions in each variable. The coefficient involved in each contribution is a constant. Its value does not depend on the particular values that the sources happen to assume. It depends only on the resistance values, structure of the circuit and the location where the particular independent source is connected. Therefore, we expect the coefficients of I_1 to remain at $\frac{3}{13}$ and $\frac{1}{13}$ in i_1 and i_2 respectively whatever be the values I_2 and V_1 happen to have. And, we choose to think of the situation when I_2 and V_1 are zero-valued. Then the circuit has only one source and it will produce

$i_1 = \frac{3}{13} I_1$ and $i_2 = \frac{1}{13} I_1$ since the coefficients are independent of source values. Similarly the circuit will have $i_1 = -\frac{1}{13} I_2$ and $i_2 = -\frac{5}{26} I_2$ when $I_1 = V_1 = 0$ and $i_1 = -\frac{2}{13} V_1$ and $i_2 = \frac{3}{26} V_1$ when $I_1 = I_2 = 0$.

Thus, we can view the solution of the circuit, when I_1, I_2 and V_1 are simultaneously acting, as the sum or *superposition* of solution of three identical circuits with only one of the sources taking a non-zero value in each circuit. *A current source with zero-value is an open-circuit and a voltage source with zero value is a short-circuit*. Hence, in the first circuit we replace I_2 by an open-circuit and V_1 by a short-circuit and solve it to get the first contribution due to I_1. In the second circuit we replace I_1 by an open-circuit and V_1 by a short-circuit and solve it to get the second contribution due to I_2. And in the third circuit we replace I_1 by a open-circuit and I_2 by an open-circuit and solve it to get the third contribution due to V_1. We add the contributions to get the solution for the original circuit in which all the three sources were acting simultaneously. *We can do this only because the contributions from various sources stand segregated in the form of a linear combination with no interaction among them.*

A multi-source circuit problem can be split up into many single-source circuit problems.

Thus, a multi-source circuit problem can be split up into many single-source circuit problems and the response for any circuit variable in the multi-source circuit can be found as the *superposition* (*i.e.,* sum) of responses for same circuit variable in all those single-source circuits. In doing so, we will be constructing the final solution by piecing together the individual contributions from independent sources.

We systematize this further. We note that the contribution from any particular independent source to a particular response variable is proportional to the source function value. We have been terming the proportionality constant as a *coefficient of contribution.* In $x = a_1I_1 + a_2I_2 + ... + b_1V_1 + b_2V_2 + ...$, where x is some circuit response variable and I_1 , I_2 , ... are the independent current source functions and V_1, V_2 , ... are the independent voltage source functions, the a's and b's are the proportionality constants or the so-called coefficients of contribution. They are the ones that matter; not the particular values of source functions. Each coefficient can be interpreted as the *contribution to the circuit variable due to unit value of a particular input source - i.e.,* contribution per unit input. If we know the *contribution per unit input* for each source, we can find out the contribution due to that source by a simple *scaling* operation that involves multiplying *contribution per unit input* by the source function value.

Now we are ready to state the different forms of *Superposition Theorem.*

Superposition Theorem Form-1

Superposition Theorem – First form

"*The response of any circuit variable in a multi-source linear memoryless circuit containing 'n' independent sources can be obtained by adding the responses of the same circuit variable in n single-source circuits with i^{th} single-source circuit formed by keeping only i^{th} independent source active and all the remaining independent sources deactivated*."

Deactivation of an independent current source is achieved by replacing it with an open-circuit and deactivation of an independent voltage source is achieved by replacing it with a short-circuit. *Dependent sources are not to be treated as sources while applying Superposition Theorem. They will be present in all the single-source component circuits.*

The principle embodied in the above can also be stated in the following manner.

Superposition Theorem Form-2

Superposition Theorem – Second form

"*The response of any circuit variable x in a multi-source linear memoryless circuit containing 'n' independent sources can be expressed as* $x(t) = \sum_{i=1}^{i=n} a_i U_i(t)$, *where $U_i(t)$ is the source function of i^{th} independent source (can be a voltage source or current source) and a_i is its 'coefficient of contribution'. The coefficient of contribution has the physical significance of contribution per unit input.*"

The coefficient of contribution, a_i, which is a constant for a time-invariant circuit, can be obtained by solving for $x(t)$ in a single-source circuit in which all independent sources other than the i^{th} one are deactivated by replacing independent voltage sources with short-circuits and independent current sources with open-circuits.

But, why should a linear combination $x = a_1I_1 + a_2I_2 + ... + b_1V_1 + b_2V_2 + ...$ be found term by term always? Can't we get it in subsets that contain more than one term? The third form of *Superposition Theorem* states that it can be done.

Superposition Theorem Form-3

Superposition Theorem – Third form

"*The response of any circuit variable in a multi-source linear memoryless circuit containing 'n' independent sources can be obtained by adding responses of the same circuit variable in two or more circuits with each circuit keeping a subset of independent sources active in it and remaining sources deactivated such that there is no overlap between the such active source subsets among them*."

Linearity of a Circuit

Linearity of a circuit element and linearity of a circuit are two different concepts.

A circuit is called *linear* if its solution obeys superposition principle. This is why we stated the Superposition Theorem with the adjective *linear* behind 'circuit'. Whether we view the statements on Superposition Theorem as a definition of linearity of a circuit or as an assertion of an important property of linear circuits is matter of viewpoint.

There is indeed a bit of circularity in *Linearity* and *Superposition Principle*.

Why did the memoryless circuits we have been dealing with till now obey superposition principle? The elements of memoryless circuits were constrained to be linear time-invariant elements. We used only *linear* resistors and *linear* dependent sources. The *v-i* relations of all those elements obey superposition principle. As a result, all KCL and KVL equations in nodal analysis and mesh analysis had the form of linear combinations. Such KVL and KCL equations lead to nodal conductance matrix (and mesh resistance matrix) that contain only constants in the case of a time-invariant circuit (*i.e.*, resistances are constants and coefficients of dependent sources are also constants). Similarly, the input matrix (***C*** in nodal analysis and ***D*** mesh analysis) will contain only constants in the case of circuits constructed using linear time-invariant elements. Thus, the solution for node voltage variables and mesh current variables will come out in the form of linear combination of independent source functions. And, after all Superposition Theorem is only a restatement of this fact. Therefore, Superposition Theorem holds in the circuit since we used only linear elements in constructing it except for independent sources which are non-linear. Hence, we conclude that *a memoryless circuit constructed from a set of linear resistors, linear dependent sources and independent sources (they are non-linear elements) results in a circuit which obeys Superposition Theorem and hence, by definition, is a linear circuit.*

Linearity of a circuit element and linearity of a circuit are two different concepts. An element is linear if its *v-i* relationship obeys principle of homogeneity and principle of additivity. A circuit is linear, if all circuit variables in it, without any exception, obey principle of homogeneity and principle of additivity, *i.e.*, the principle of superposition. It may appear intuitively obvious that a circuit containing only linear elements will turn out to be a linear circuit. But remember that we did use non-linear elements - independent sources are non-linear elements. Then, it is not so obvious. The above discussion offers a plausibility reasoning to convince us that a circuit containing linear elements and independent sources will indeed be a linear circuit. But the mathematical proof for this apparently straightforward conclusion is somewhat formidable.

Linearity and Superposition appear so natural to us. But the fact is that most of the practical electrical and electronic circuits are non-linear in nature. Linearity, at best, is only an approximation that circuit analysts employ to make the analysis problem more tractable. We illustrate why Superposition Theorem does not hold for a circuit containing a non-linear element by an example. See circuit (a) in Fig. 5.1-3 . The resistor R in it is a non-linear one with a *v-i* relation given by $v = 2i^2$ for $i \geq 0$ and $-2i^2$ for $i < 0$.

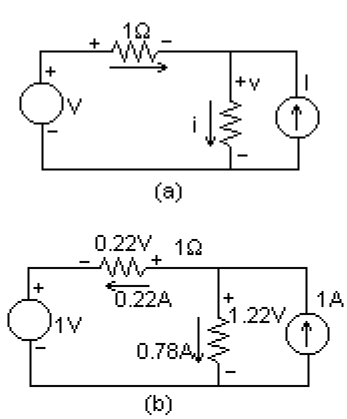

Fig. 5.1-3 (a) A Circuit Containing a Non-linear Resistor (b) Circuit Solution for V = 1 volt and I = 1 amp

The circuit is solved by writing the KVL equation in the first mesh. We first make use of KCL at the current source node to obtain the current through the 1Ω resistor as i-I amps. Then, KVL in the first mesh gives

$$-V+(i-I)+2i^2 = 0 \;\Rightarrow\; i = 0.25[\sqrt{1+8(V+I)}-1] \text{ amps}$$

The value of this current for V = 1 volt and I = 1 amp is 0.78 amps. Corresponding voltage across the non-linear resistor is = $2i^2 \approx 1.22$ V and the remaining circuit variables can now be obtained easily. The complete solution is marked in circuit (b) of Fig. 5.1-3.

We find out the circuit solution when the independent sources are acting one by one. Fig. 5.1-4 show the relevant sub-circuits and solution.

The circuit (a) is solved by using the KVL equation $-V + i + 2i^2 = 0$. The solution for i will be $i = 0.25[\sqrt{1+8V} - 1]$ amps . The solution for a case with V = 1volt and I = 1amp is marked in circuit (b) of Fig. 5.1-4.

The circuit (c) is solved KCL at the current source node $2i^2 + 2i - I = 0$. The solution for i will be $i = 0.5[\sqrt{1+2I} - 1]$ amps. The solution for a case with V = 1volt and I = 1amp is marked in circuit (d) of Fig. 5.1-4.

We observe that the current through the non-linear resistor when both sources are acting simultaneously is 0.78 A whereas the sum of responses from two circuits (circuit (a) and (c) of Fig. 5.1-4) is 0.5 A + 0.37 A = 0.87 A. Thus Superposition does not work in this circuit.

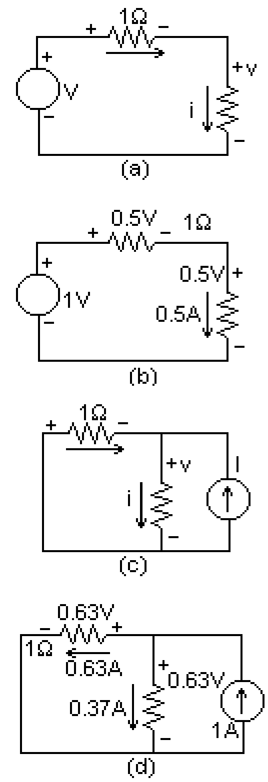

Fig. 5.1-4 Circuits with one independent source acting at a time and circuit solution

In general, $0.25[\sqrt{1+8(V+I)} - 1] \neq 0.25[\sqrt{1+8V} - 1] + 0.5[\sqrt{1+2I} - 1]$ and hence this circuit does not obey Superposition Theorem. We also note that it is not possible to identify the contributions from the independent voltage source and independent current source separately when the two sources are acting simultaneously. We may try expanding the $\sqrt{1+8(V+I)}$ term in the solution for i in binomial series. Then we get,

$$i = [(V+I) - 0.25(V+I)^2 + \cdots\cdots] = V + I - 0.25V^2 - 0.25I^2 - 0.5VI + \cdots$$

Thus, i is decided by V and I through their higher powers along with first power terms. Higher power terms can not satisfy superposition principle. Moreover, there are cross product terms like VI, V^2I, VI^2 etc., in the expression. We can not ascribe such terms to voltage source or current source exclusively. We may take the view that they are the contributions from current source. In that case we have to admit that the contribution from the current source to the current i depends on whether the other source is active or not. And that kind of dependence results in non-adherence to superposition principle. Thus, we conclude that, non-linear elements in a circuit results in the circuit response failing to meet superposition principle due to (i) independent sources contributing to response variables through their higher powers and (ii) independent sources contributing jointly to response variables through cross product terms.

Example : 5.1-1

An independent voltage source and an independent current source from outside drive a memoryless circuit containing no independent sources within it as shown in Fig. 5.1-5. The current delivered by the voltage source is found to be 1 A when the current source is disconnected, 2 A when the current source is delivering 10 A into the circuit and 0.5 A when the current source is taking out 10 A from the circuit. Is the memoryless circuit a linear one?

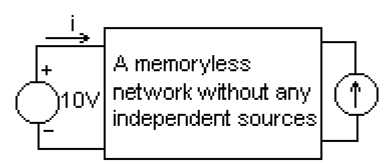

Fig. 5.1-5 Circuit for Example : 5.1-1

Solution

Let us assume that the circuit is linear. Then the current delivered by the 10 V source ca be expressed as a linear combination of the two source function.

i.e., $i = a \times 10 + bI$. The source function value of voltage source has been substituted in this equation. i is given to be 1 A when I = 0 and 2 A when I = 10 A. Therefore a = 0.1 amp/volt and b = 0.1 amp/amp if the circuit is linear.

Then, when I = -10 A, the current delivered by the voltage source must be 0.1×10-0.1×10 = 0 A. But it is stated that the current observed under this condition is 0.5A.

Hence the circuit within the box is not linear.

Example : 5.1-2

A certain resistor R in a linear memoryless network driven by two independent sources as shown in Fig. 5.1-6 is found to dissipate 36 W when only the voltage source is acting and 64 W when only the current source is acting. Find the power dissipated in the resistor when both sources are acting simultaneously. Is the answer unique?

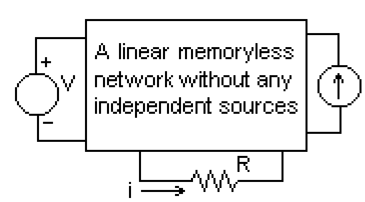

Fig. 5.1-6 Circuit for Example : 5.1-2

Solution

Since the network is linear the current through the resistor R can be expressed as a linear combination of V and I.

$\therefore i = aV + bI$

The power dissipated, *i.e.*, i^2R is given as 36 W when $I = 0$ and 64 W when $V = 0$.

$\therefore P_v = (aV)^2 R = 36$ and $P_i = (bI)^2 R = 64$

$\therefore (aV) = \sqrt{P_v / R}$ and $(bI) = \sqrt{P_i / R}$

The power that will be dissipated when both sources are acting simultaneously is given by

Power is not a superposable quantity

$$\begin{aligned} P_{vi} &= (aV + bI)^2 R \\ &= (aV)^2 R + (bI)^2 R + 2(aV)(bI)R \\ &= P_v + P_i + 2\sqrt{P_v / R}\sqrt{P_i / R}\, R \\ &= P_v + P_i + 2\sqrt{P_v P_i} = 36 + 64 + 2\sqrt{36 \times 64} = 196\text{W} \end{aligned}$$

Note that the power dissipated when both sources are acting is not the sum of powers dissipated when one source is acting at a time. *i.e., power is not a superposable quantity*. The reason is very simple – $(i_1 + i_2)^2 \neq i_1^2 + i_2^2$ and $(v_1 + v_2)(i_1 + i_2) \neq v_1 i_1 + v_2 i_2$ are the reasons.

The power calculated as 196 W is not a unique answer. Power dissipated in a resistor when a certain current is flowing through it is independent of direction of the current since power depends on square of the resistor current. We have unconsciously assumed that both a and b are positive or negative. But we have to account for the possibility of a and b having opposite signs - *i.e.*, the possibility of two current contributions canceling each other partially. This possiblity is taken into account by modifying the total power equation as $P_{vi} = P_v + P_i \pm 2\sqrt{P_v / R}\sqrt{P_i / R}\, R$. Hence the second possible value of power when both sources are acting simultaneously is 4 W.

Additional information in the form of current values or voltage values will be needed to decide between 196 W and 4 W.

Example : 5.1-3

The source function values for the three independent sources in Fig. 5.1-7 are V= 10 V, I_1 = 1 A and I_2 = 2 A. The current in the resistor R is seen to be 1 A when the two current sources are switched off and 1.5 A when only I_2 is switched off and 2 A when all the three sources are active. Find what voltage must be applied by the voltage source if the current in R is to become zero with no change in current source values?

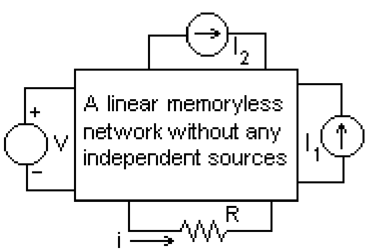

Fig. 5.1-7 Circuit for Example : 5.1-3

Solution

The current through R can be written as a linear combination of three source functions.

i.e., $i = aV + bI_1 + cI_2$

Substituting the data stated in the problem we get three equations in three unknowns as below.

$1 = 10a$

$1.5 = 10a + b$

$2 = 10a + b + 2c$

Solving this system of equations, $a = 0.1$, $b = 0.5$ and $c = 0.25$.

$\therefore i = 0.1V + 0.5I_1 + 0.25I_2$

The required voltage to make i zero with $I_1 = 1$ A and $I_2 = 2$ A is obtained by

$0 = 0.1V + 0.5 + 0.5 \Rightarrow V = -10$ volts

Example : 5.1-4

Find the value of *V* in the circuit in Fig. 5.1-8 such that the voltage source delivers zero power to the circuit by using Superposition Theorem.

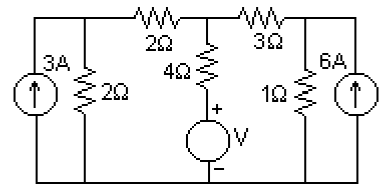

Fig. 5.1-8 Circuit for Example : 5.1-4

Solution

The circuit is a linear one. Therefore, the current delivered by the voltage source can be expressed as a linear combination of the three source functions. Power delivered by the voltage source will be zero if the current delivered by it is zero. Thus, we want a value of *V* such that $aV + bI_1 + cI_2 = 0$ where *a, b* and *c* are the per unit contributions to current delivered by the voltage source from the three source functions.

We determine the three contributions first by solving three single-source circuits shown in Fig. 5.1-9.

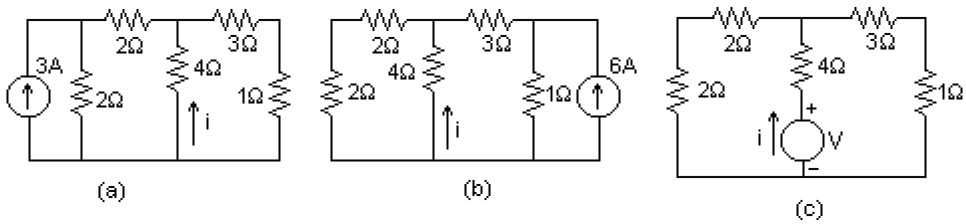

Fig. 5.1-9 Sub-circuits for Applying Superposition Theorem in Example : 5.1-4

The value of *i* in circuit (a) is found as $-3\text{A} \times \frac{2}{2+(2+4//(3+1))} \times \frac{3+1}{4+(3+1)} = -0.5\text{A}$

The value of *i* in circuit (b) is found as $-6\text{A} \times \frac{1}{1+(3+4//(2+2))} \times \frac{2+2}{4+(2+2)} = -0.5\text{A}$

The value of *i* in circuit (c) is found as $\frac{V}{4+(2+2)//(3+1)} = \frac{V}{6}\text{A}$

$\therefore$Current delivered by voltage source when all the three sources are active =

$$= -0.5 - 0.5 + \frac{V}{6} = \frac{V-6}{6}\text{A}$$

$\therefore$The value of *V* such that the current (and power) delivered by the voltage source will be zero = 6 V.

We note that we did not have to resort to node analysis or mesh analysis in solving the three single-source circuits shown in Fig. 5.1-9.

Example : 5.1-5

Find the power dissipated in the resistor R_2 in the circuit in Fig. 5.1-10 by applying Superposition Theorem.

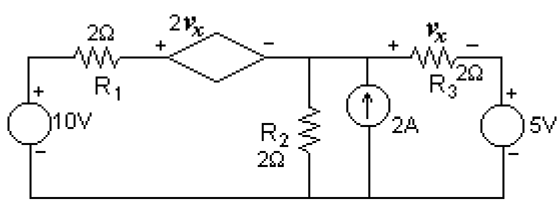

Fig. 5.1-10 Circuit for Example : 5.1-5

Solution

We find the current through R_2 by applying Superposition Theorem first. The single-source circuit when the first voltage source is acting is shown in Fig. 5.1-11.

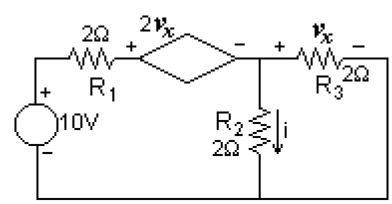

Fig. 5.1-11 Circuit with only the first voltage source acting in Example : 5.1-5

This circuit is solved by KVL in the first mesh. Let the first mesh current be i_1. Then $v_x = 2//2\ i_1 = i_1$ itself.

$-10+2i_1+2i_1+i_1 = 0 \Rightarrow i_1 = 2\text{A}$ and hence $i = 1$ A

The single-source circuit when the second voltage source is acting alone is shown in circuit (a) of Fig. 5.1-12.

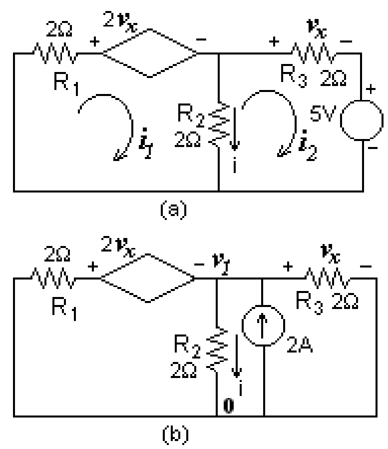

Fig. 5.1-12 Circuits with only (a) the second voltage source acting and (b) only the current source is acting in Example : 5.1-5

This circuit is solved by mesh analysis. The controlling variable v_x of the dependent source is $2i_2$. The mesh equations are

$$-2i_1 + 4i_2 + 2(i_1 - i_2) = 0$$

$$2(i_2 - i_1) + 2i_2 + 5 = 0$$

Simplifying the equations lead to $2i_1 + i_2 = 0$ and $-2i_1 + 4i_2 = -5$. Solving these we get $i_1 = 0.5$ A and $i_2 = -1$ A. Therefore $i = i_1 - i_2 = 1.5$ A.

The circuit (b) in Fig. 5.1-12 shows the single-source circuit when only the current source is acting. This circuit is solved by nodal analysis. The node voltage v_1 is assigned as shown in circuit diagram. The controlling variable of dependent source is same as v_1. Therefore, the potential at the right end of R_1 is $3v_1$ with respect to the reference node. Writing KCL at the top node, $0.5v_1 + 0.5v_1 + 0.5\times3v_1 = 2 \Rightarrow 2.5\,v_1 = 2 \Rightarrow v_1 = 0.8$ volts. Therefore $i = 0.4$ amps.

Therefore the current in R_2 when all the three sources are acting simultaneously is = 1+1.5+0.4 = 2.9 A

Therefore the power dissipated in $R_2 = 2\times2.9^2 = 16.82$ W.

5.2 Star-Delta Transformation Theorem

We observe from the examples on application of Superposition Theorem in the last section that the single-source circuits that need to be solved in that context may require us to use nodal analysis and mesh analysis often. However, we can expect to avoid these procedures in the case of circuits involving only resistors and independent sources. We will be able to solve the single-source circuits by employing series and parallel equivalents repeatedly. But there is one pair of resistor connections that will not yield to this kind of approach. For instance, consider the problem of finding the current through the resistor R in circuit (a) of Fig. 5.1-13 by applying superposition principle. The relevant single-source circuits are shown in (b) and (c) of the same figure.

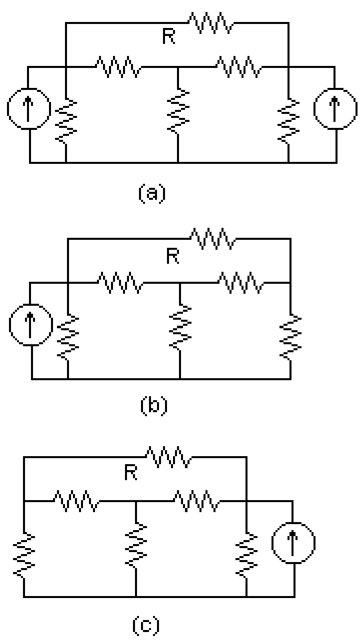

Fig. 5.1-13 A Circuit in which Start-Delta Transformation will be helpful

This problem can not be solved by series-parallel equivalents. The T-shaped (also called Y-shaped or Star-connected) resistor network containing three resistors makes it impossible to apply series-parallel reduction. Equivalently, the three outer resistors which are connected in Π form (also called Delta-connected, Mesh-connected etc.) makes it impossible to apply series-parallel reduction.

It turns out that an Y-connected set of three resistors can be replaced with a Δ-connected set of three resistors without any circuit variable outside these three resistors getting affected. Similarly, a set of three resistors connected in Δ can be replaced with a set of resistors connected in Star without affecting the circuit solution in the remaining portion of the circuit. We develop equations for this transformation in this section.

First we consider Star to Delta Transformation. We want the two resistor networks shown in Fig. 5.2-1 to be equivalent with respect the external network.

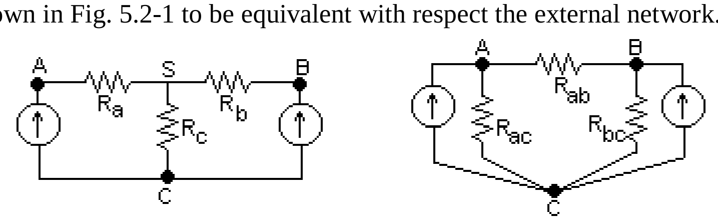

Fig. 5.2-1 Circuits Related to Star-Delta Transformation

The logic behind Star-Delta transformation

The net effect of the external network on the star connected resistor may be modeled by two current sources driving it as shown. If the second network that is connected in delta produces same node voltages at node-A and node-B with respect to node-C when driven by the same two current sources as in the star network, the external circuit solution will not be affected in any sense. This is so since node voltage variables in a circuit decide all other voltages and currents in a circuit. If the node voltage variables do not get affected, then, no voltage or current in the circuit gets affected. Therefore, we can derive the values for resistors in delta network in terms of resistor values in star network by imposing the condition that v_A and v_B with respect to node-C must be the same in both circuits when driven by same current sources.

Let I_1 and I_2 be the current source functions. Then v_A and v_B in star circuit is given by

$$\begin{bmatrix} G_a & 0 & -G_a \\ 0 & G_b & -G_b \\ -G_a & -G_b & G_a+G_b+G_c \end{bmatrix}\begin{bmatrix} v_A \\ v_B \\ v_S \end{bmatrix}=\begin{bmatrix} I_1 \\ I_2 \\ 0 \end{bmatrix}$$

We do not need the node voltage at the node-S. We eliminate it easily since there is no current source injection at that node.

$$-G_a v_A - G_b v_B + (G_a+G_b+G_c)v_S = 0$$
$$\therefore v_S = \frac{G_a}{G_a+G_b+G_c}v_a + \frac{G_b}{G_a+G_b+G_c}v_b$$

Substituting the expression for v_S in the first two node equations, rearranging terms and expressing the final equations in matrix form, we get,

$$\begin{bmatrix} \frac{G_a(G_b+G_c)}{(G_a+G_b+G_c)} & \frac{-G_aG_b}{(G_a+G_b+G_c)} \\ \frac{-G_aG_b}{(G_a+G_b+G_c)} & \frac{G_b(G_a+G_c)}{(G_a+G_b+G_c)} \end{bmatrix}\begin{bmatrix} v_A \\ v_B \end{bmatrix}=\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad (5.2\text{-}1)$$

Now we write the node equation for the delta-connected network.

$$\begin{bmatrix} G_{ac}+G_{ab} & -G_{ab} \\ -G_{ab} & G_{ab}+G_{bc} \end{bmatrix}\begin{bmatrix} v_A \\ v_B \end{bmatrix}=\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad (5.2\text{-}2)$$

Both the equations, Eqn. 5.2-1 and Eqn. 5.2-2 should result in same v_A and v_B for network equivalence. This requires that the two nodal conductance matrices be equal. Therefore,

$$G_{ab} = \frac{G_aG_b}{G_a+G_b+G_c}$$
$$G_{ac} = \frac{G_a(G_b+G_c)}{G_a+G_b+G_c} - \frac{G_aG_b}{G_a+G_b+G_c} = \frac{G_aG_c}{G_a+G_b+G_c} \quad (5.2\text{-}3)$$
$$G_{bc} = \frac{G_b(G_a+G_c)}{G_a+G_b+G_c} - \frac{G_aG_b}{G_a+G_b+G_c} = \frac{G_bG_c}{G_a+G_b+G_c}$$

Expressing this in terms of resistances,

$$R_{ab} = \frac{R_aR_b + R_bR_c + R_cR_a}{R_c}$$
$$R_{ac} = \frac{R_aR_b + R_bR_c + R_cR_a}{R_b} \quad (5.2\text{-}4)$$
$$R_{bc} = \frac{R_aR_b + R_bR_c + R_cR_a}{R_a}$$

Expressions for determining Delta network resistors from Star network resistors.

Eqn. 5.2-4 shows how the resistance values for equivalent delta may be calculated from resistance values used in star network.

Fig. 5.2-2 shows the star-delta transformation in a way that makes the symmetry in the equations evident.

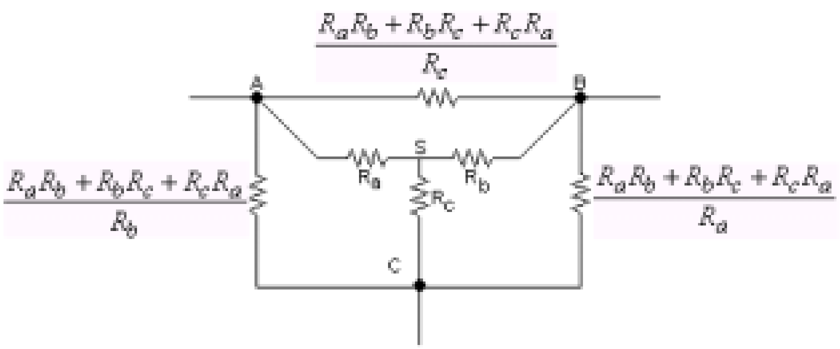

Fig. 5.2-2 Star to Delta Transformation Equations

The Delta-Star transformation may similarly be derived using an approach based on mesh analysis. Assume that a pair of independent voltage sources drives both circuits. Then the currents drawn from the sources must be the same in both circuits. The delta network will have three meshes; however the central mesh has no voltage source in it and hence its mesh current can be eliminated by using the technique we employed in this section. The details of this derivation are skipped and the final result is given as

Expressions for determining Star network resistors from Delta network resistors

$$R_a = \frac{R_{ab}R_{ac}}{R_{ab}+R_{bc}+R_{ac}},\ R_b = \frac{R_{ab}R_{bc}}{R_{ab}+R_{bc}+R_{ac}},\ R_c = \frac{R_{ac}R_{bc}}{R_{ab}+R_{bc}+R_{ac}} \quad (5.2\text{-}5)$$

Eqn. 5.2-5 shows how the resistance values of the equivalent star network may be obtained from the delta network resistances and Fig. 5.2-3 shows this in a manner that makes the symmetry in these equations evident.

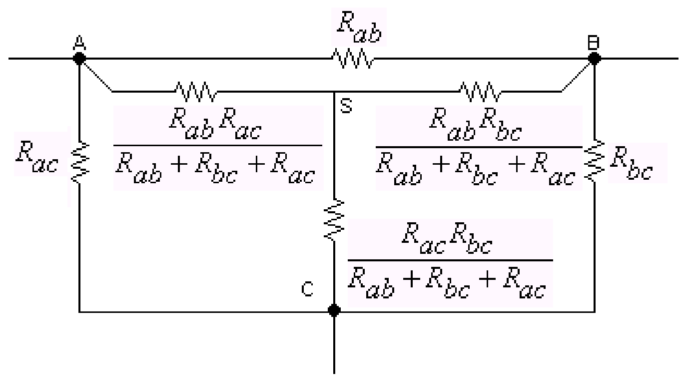

Fig. 5.2-3 Delta to Star Transformation Equations

Star-Delta (also called *Y-Δ* or *T-Π* transformation) transformation is widely employed in analysis of three-phase ac circuits. A case of special interest is that of equal resistances in all limbs of star or delta. The transformation equations for this special case are shown in Fig. 5.2-4 .

A special case of equal resistors in Delta or Star

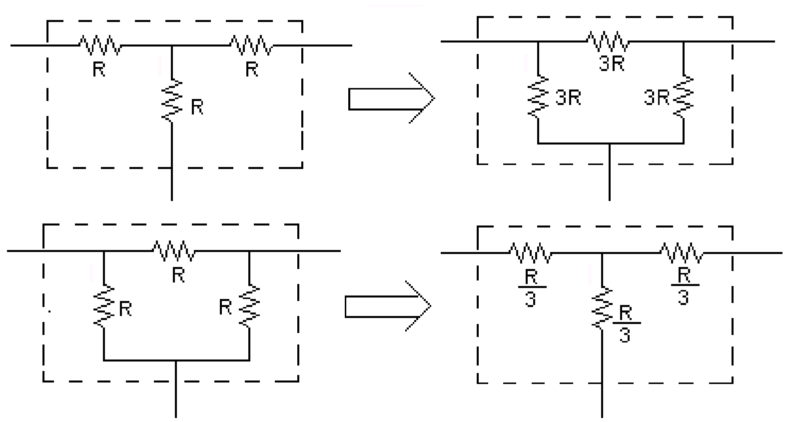

Fig. 5.2-4 A Special Case of Star-Delta Transformation and Delta-Star Transformation

Example : 5.2-1

Find the power dissipated in the resistor *R* in circuit in Fig. 5.2-5 .

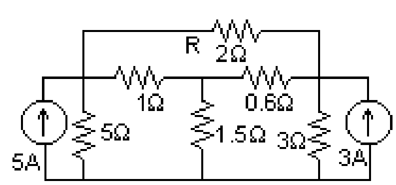

Fig. 5.2-5 Circuit for Example : 5.2-1

Solution

The single-source circuits required for applying superposition principle are shown in (a) and (c) of Fig. 5.2-6.

Circuits (b) and (d) show the circuits (a) and (b) respectively after star-delta transformation on the inner star-connected resistors of 1Ω , 0.6Ω and 1.5Ω. Taking positive direction of current flow to be from left to right in *R*, the current flowing in *R* in circuit (b) is $5\,\text{A}\times\frac{5//5}{5//5+(2//2+3//3)}\times\frac{2}{2+2}=1.25\,\text{A}$. Similarly the current flowing in it in circuit (d) is $-3\,\text{A}\times\frac{3//3}{3//3+(2//2+5//5)}\times\frac{2}{2+2}=-0.45\,\text{A}$.

Therefore the current through *R* when both sources are acting will be 1.25-0.45 = 0.8 A.

Therefore power dissipation in $R = 0.8^2 \times 2 = 1.28$ W

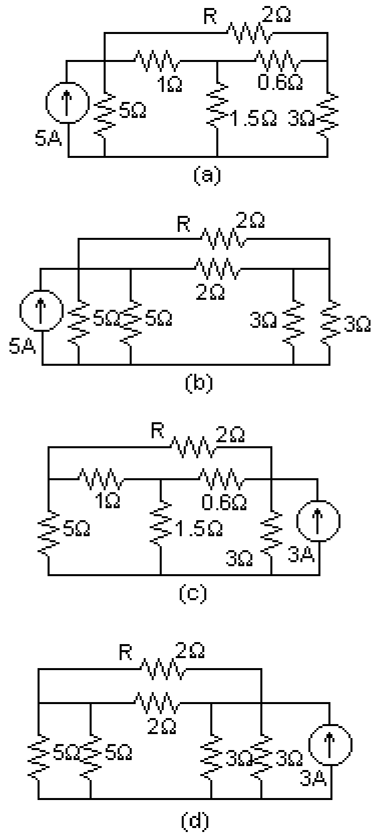

Fig. 5.2-6 Single-source Circuits for Applying Superposition Theorem in Example : 5.2-1

5.3 Substitution Theorem

Consider the three-mesh circuit shown in Fig. 5.3-1. Mesh analysis reveals that the mesh currents are 1A, 2A and 3A as shown in the figure. Two nodes **a** and **a’** have been identified in the circuit and the current crossing the node **a** from left to right is marked as 2A. The voltage of **a** with respect to **a’** is calculated to be 1 V and is marked in the figure.

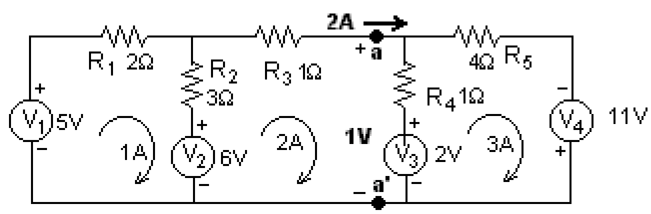

Fig. 5.3-1 A Three-mesh Circuit with Two Nodes - **a** and **a'** - Identified

Now we add two current sources between the two nodes, **a** and **a’**, as shown in Fig. 5.3-2. The current sources have equal and opposite currents of 2 A magnitude.

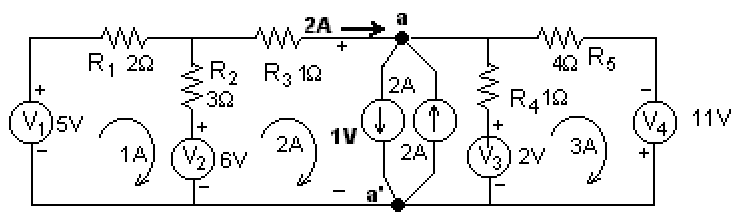

Fig. 5.3-2 Circuit in Fig. 5.3-1 with two current sources added

We have not changed the KCL equation at node **a** and node **a’**. The mesh introduced in this step is a trivial mesh. Hence the circuit solution everywhere will remain the same as before. Now introduce a pair of nodes **b** and **b’** that are connected to **a** and **a’** respectively by shorting links as in Fig. 5.3-3.

Application of KCL at node **a** and **a’** show that there is no current flow in the two shorting links.

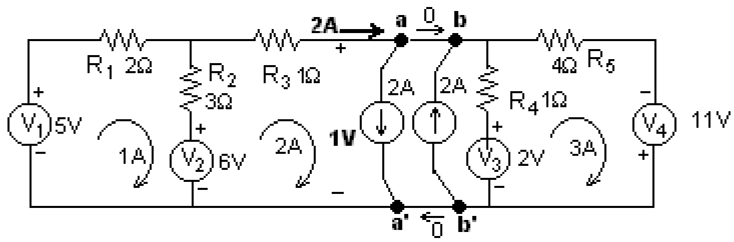

Fig. 5.3-3 Additional Node pair **b** and **b'** Identified in Circuit of Fig. 5.3-1

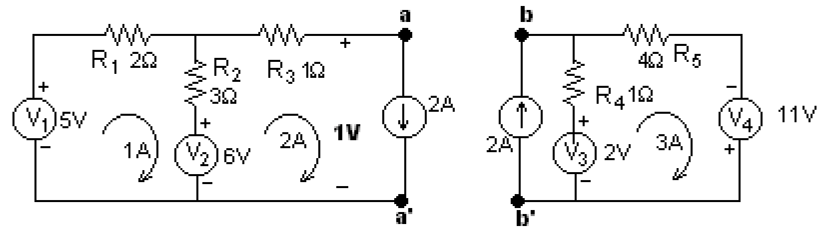

Fig. 5.3-4 The Original Circuit separated into two parts without the solution in either part getting affected

We note that the current flows in the shorting link **a-b** and **a'-b'** are zero. Therefore, breaking the shorting links should not affect the circuit solution in both parts of the original circuit. See Fig. 5.3-4.

Thus, as far as the first part is concerned, we have been able to replace or *substitute* the second part with a current source, which has a value exactly equal to the current drawn by the second part of the circuit from first part of the circuit, without any circuit variable in the first part undergoing any change. Similar statement can be framed for second part of the circuit too.

Now, let us go back to Fig. 5.3-1 and add two independent voltage sources instead of current sources as shown in Fig. 5.3-5. We have not affected the KVL in second mesh in any way and hence circuit solution remains the same as before everywhere. However, we have reduced the voltage between **a** and **a'** to zero.

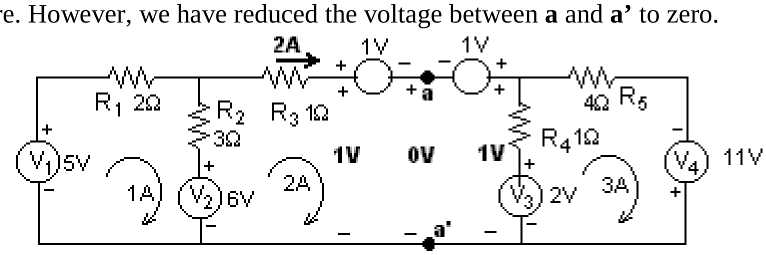

Fig. 5.3-5 Circuit in Fig. 5.3-1 with two equal and opposite voltage sources introduced in series at node **a**

If **a** and **a'** are at same potential, they can be joined together. If they can be joined together, the two parts of the circuit are connected at only one point and hence they can not affect each other in any way. We can very well draw them as separate circuits without a common touch point as shown in Fig. 5.3-6.

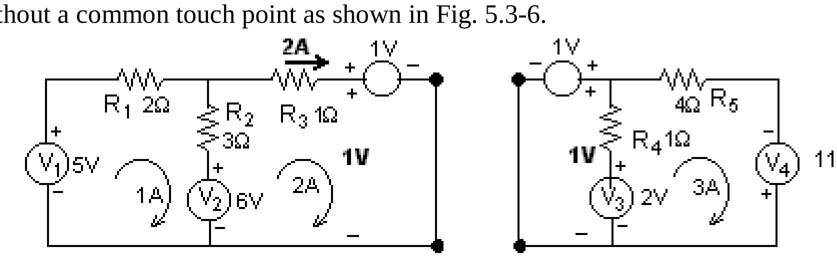

Fig. 5.3-6 Original Circuit is separated into two parts without the circuit solution in either part getting affected

Thus, as far as the first part is concerned, we have been able to replace or *substitute* the second part with a voltage source, which has a value exactly equal to the voltage that was impressed on the second part of the circuit by the first part of the circuit, without any circuit variable in the first part undergoing any change. Similar statement can be framed for second part of the circuit too.

We go one step further and end up in trouble! We extract the first part from Fig. 5.3-4 and apply the same reasoning we employed to arrive at the two parts in that figure to arrive at the circuit shown in (b) of Fig. 5.3-7.

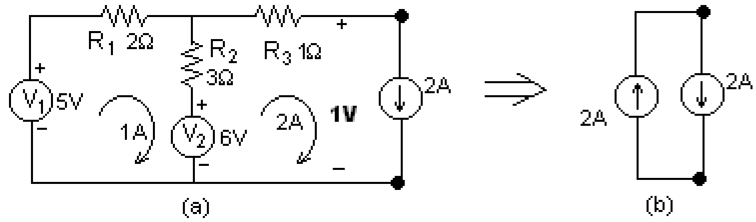

Fig. 5.3-7 The result of stretching an idea too much!

That circuit in Fig. 5.3-7(b) has no unique solution since the voltage across the current sources can now be any value without violating any circuit laws.

Thus, there must be some constraints to be satisfied by a circuit if *substitution* of a part of the circuit by a current source of value equal to the current drawn by it (or by a voltage source of value equal to voltage appearing across it) is not to affect the circuit solution in the remaining part. There are. *The constraint is that the original circuit must have a unique solution and the circuit after substitution also must have a unique solution.* Linear circuits usually have unique solution –*i.e.*, the currents and voltages everywhere are *uniquely* decided by values of independent sources and the circuit structure – except in some trivial and avoidable situations like ideal independent voltage sources in parallel or ideal independent current sources in series etc.

A constraint to be satisfied by a circuit so that Substitution Theorem can be applied to it.

However, note that we used only KCL and KVL based arguments to arrive at the validity of *substitution*. We did not make use of element relations at all. Hence, the arguments are valid for any circuit – linear or non-linear. *Substitution Theorem* is more general than *Superposition Theorem*. The constraint of *unique solution* assumes particular significance in the case of non-linear circuits since there are non-linear that have multi-valued *v-i* relationships. A tunnel diode, an uni-junction transistor etc. are some examples.

There is another constraint to be satisfied before *substitution* can be done in a circuit. Consider the situation where the controlling variable of a dependent source is in the part that was subjected to substitution with the dependent source output connected in the other part. Obviously that will not work. Therefore, if there are dependent sources in the part of the circuit that is being substituted by an independent current source or voltage source, both the controlling variable and the dependent source must be within that part of the circuit. Similarly, if there is magnetic coupling in the part of the circuit being substituted, all coils belonging to the magnetically coupled system must be within that part of the circuit. This constraint may alternatively be stated as – *there should not be any interaction between the part of the circuit that is being substituted and the remaining circuit except through the pair of terminals at which they are interconnected.*

Another constraint to be satisfied by a circuit so that Substitution Theorem can be applied to it.

Subject to the constraints on *unique solution* and *interaction only through the connecting terminals*, we state the *Substitution Theorem* as below.

Let a circuit with unique solution be represented as interconnection of two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 and N_2 may be linear or non-linear. Let v(t) be the voltage that appears at the terminals between N_1 and N_2 and let i(t) be the current flowing into N_2 from N_1. Then, the network N_2 may be replaced by an independent current source of value i(t) connected across the output of N_1 or an independent voltage source of value v(t) connected across the output of N_1 without affecting any voltage or current variable within N_1 provided the resulting network has unique solution.

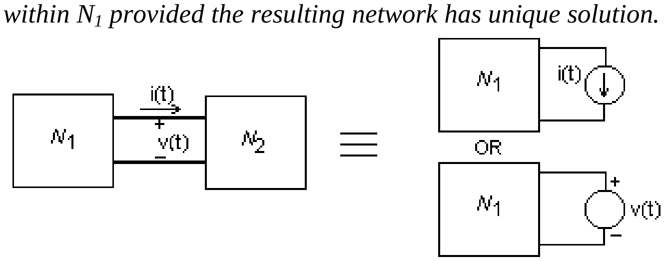

Fig. 5.3-8 The Substitution Theorem

But, what is the use of a theorem that wants us to solve a circuit first and then replace part of the circuit by a source that has a value depending on the solution of the circuit? Obviously, such a theorem will not help us directly in solving circuits.

The significance of this theorem lies in the fact that it can be used to construct theoretical arguments that lead to other powerful circuit theorems that indeed help us to solve circuit analysis problems in an elegant and efficient manner.

Moreover, it does find application in circuit analysis in a slightly disguised form. We take up that disguised form of Substitution Theorem in Section 5.4.

5.4 Compensation Theorem

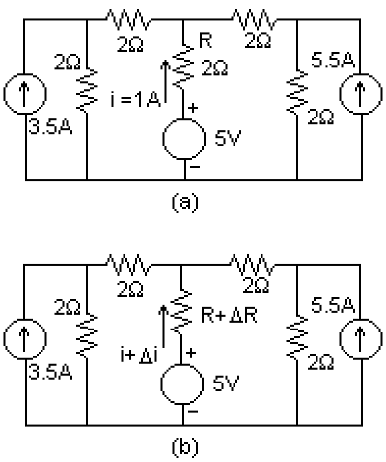

Fig. 5.4-1 Circuit to Illustrate Compensation Theorem

The circuit (a) in Fig. 5.4-1 has a resistor marked as *R*. It has a nominal value of 2Ω. Mesh analysis was carried out to find the current in this resistor and the current was found to be 1 A as marked in circuit (a) of Fig. 5.4-1.

Now assume that the resistor value changes by ΔR to $R+\Delta R$. Correspondingly all circuit variables change by small quantities as shown in circuit (b) in the same figure. The current through that resistor will also change to $i+\Delta i$. We can conduct a mesh analysis once more and get the new solution. However, we can do better than that. We can work out *changes* in variables everywhere by solving a single-source circuit and then construct the circuit solution by adding *change* to the initial solution value.

We apply Substitution Theorem on the first circuit with *R* as the element that is being substituted and on the second circuit with $R+\Delta R$ as the part that is being substituted by an independent voltage source. The voltage source in the first circuit must be *Ri* volts and the voltage source in the second circuit must be $(R+\Delta R)(i+\Delta i)$ volts.

$(R+\Delta R)(i+\Delta i) = Ri + (R+\Delta R)\Delta i + i\Delta R$. See Fig. 5.4-2.

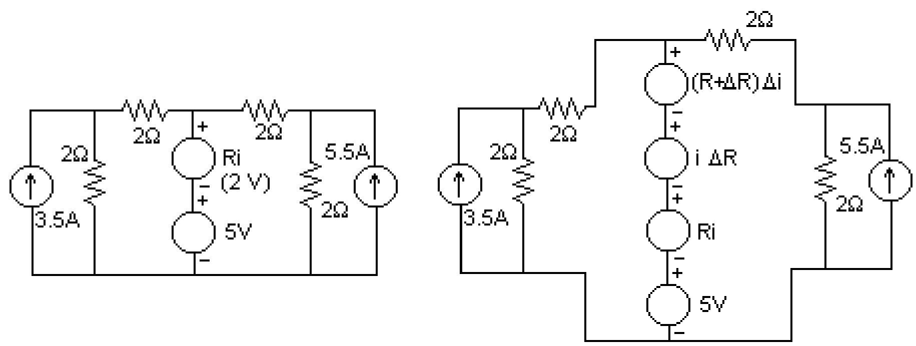

Fig. 5.4-2 Circuits after Applying Substitution Theorem

Now we solve the second circuit by applying superposition principle by taking the 5V and *Ri* volts sources along with the two current sources together first and deactivating the remaining two voltage sources. The solution we get will be the same as the solution of the original circuit since the second circuit with the $i\Delta R$ volts source and the $(R+\Delta R)\Delta i$ volts source deactivated is the same as the first circuit. We already know the solution. It is the initial solution.

We have to solve the circuit with the two sources – the $i\Delta R$ volts source and the $(R+\Delta R)\Delta i$ volts source to get the second component of complete solution for second circuit. This circuit is shown in (a) of Fig. 5.4-3. The solution of this circuit must give the *changes* in all circuit variables due the change in *R* since the initial values of variables are given by the solution contributed by the other sources. Therefore the current through central branch in the circuit (a) in Fig. 5.4-3 must be Δi.

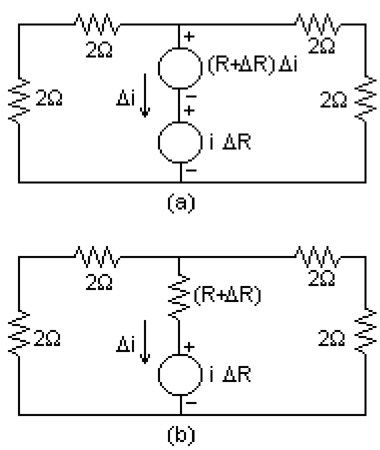

Fig. 5.4-3 (a) Circuit for obtaining changes in variables (b) After replacing voltage source by resistor

We note that the voltage of the voltage source $(R+\Delta R)\Delta i$ in circuit (a) in Fig. 5.4-3 is exactly the same as the voltage drop that will be produced by a resistor of value $(R+\Delta R)$ since the current in that branch is Δi. That is, the voltage source of value $(R+\Delta R)\Delta i$ can be thought of as the result of a substitution operation on a resistor of value $(R+\Delta R)$ in that path. We reverse this substitution and replace the voltage source by the resistor in (b) of Fig. 5.4-3. Solving circuit (b) will give us the *change* in all circuit variables due to a change in *R*. Adding the initial values to change values will give us the final solution. The circuit (b) in Fig. 5.4-3 is a single-source circuit with only one voltage source of value = (change in component value)×(initial current through that component).

Let us assume that ΔR = 0.1Ω. Then the source value is 0.1Ω×1A = 0.1 volts. Therefore,

$\Delta i = -\frac{0.1}{2.1+(2+2)//(2+2)} = -0.0244$ A and $(i+\Delta i)$ = (1-0.0244)A = 0.9756 A.

Reader may note that we used Superposition Theorem along with Substitution Theorem to arrive at this result. Hence Compensation Theorem is a specialised form of Substitution Theorem for a Linear Circuit.

Compensation Theorem

In a linear memoryless circuit, the change in circuit variables due to change in one resistor value from R to R+ΔR in the circuit can be obtained by solving a single-source circuit analysis problem with an independent voltage source of value iΔR in series with R+ΔR where i is the current flowing through the resistor before its value changed. See Fig. 5.4-4.

Compensation Theorem stated here is a specialised form of Substitution Theorem for Linear Circuits

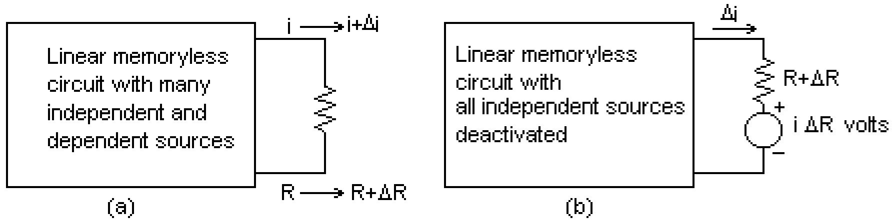

Fig. 5.4-4 The Compensation Theorem

The theorem can be extended to include dependent source coefficients too. Changes in circuit variables due to simultaneous changes in many circuit parameters can be obtained by repeated application of Compensation Theorem or as a solution of a multi-source *change* circuit in which each parameter change is taken into account by a voltage source of suitable value.

5.5 Thevenin's Theorem and Norton's Theorem

The problem of solving a circuit with different load networks connected to same delivery network arises in Electrical and Electronics Engineering quite often. We do not want to write the same node equations or mesh equations of the delivery network whenever the load network undergoes some change and solve the circuit in its entirety again and again. Thevenin's Theorem and Norton's Theorem help us to avoid this kind of wasted effort and become efficient in solving circuits. They help us to conduct node analysis or mesh analysis of the delivery network once and for all and replace it with a simple equivalent circuit for further analysis when different load networks are connected to it. They are two tools indispensable to a circuit analyst.

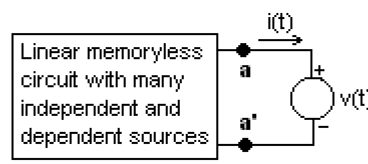

Fig. 5.5-1 A Memoryless Network Terminated in a Voltage Source at its Output Terminals

Consider a memoryless network shown in Fig. 5.5-1 containing linear resistors, linear dependent sources and independent sources with a pair of terminals identified as the output terminals of the network. The network interacts with the external world only through this pair of terminals. No parameter inside the circuit changes, but different load networks may get connected to the circuit at its output terminals. Assume that an independent voltage source of source function $v(t)$ is connected across the output terminals of the network. With no loss of generality, we assume further that the voltage source negative terminal, *i.e.*, **a'**, is taken as the reference node for writing the node equations of the circuit. We are interested in the behaviour the current $i(t)$ delivered by the circuit to the terminating voltage source versus the source function $v(t)$.

We remember that any circuit variable in a linear circuit can be expressed as a linear combination of all the independent source functions in the circuit. Hence $i(t)$ in this circuit can be expressed as

$$i(t)=[(a_1 v_{s1}(t)+a_2 v_{s2}(t)+\cdots+a_{n_v} v_{sn_v}(t))+(b_1 i_{s1}(t)+b_2 i_{s2}(t)+\cdots+b_{n_i} i_{sn_i}(t))]+a_o v(t)$$

We are applying Superposition Theorem here.

This current has two components – one contributed by all independent current sources and voltage sources within the circuit and the second contributed by the independent voltage source connected from outside, *i.e.*, $v(t)$. The functions $v_{S1}(t)$ .... represent the source functions of independent voltage sources within the circuit and the functions $i_{S1}(t)$....... represent the source functions of independent current sources within the circuit. n_v and n_i are the number of independent voltage sources and current sources within the circuit. The contribution coefficients a_o, a_1, a_2 .....and b_1, b_2, ... etc. are determined by the circuit parameters. They may be found by node analysis or mesh analysis.

The source functions and contribution coefficients are fixed once and for all by the circuit and the only aspect of the circuit that can change is the network that gets connected at the output terminals. Hence, we may represent the terms within the square brackets in the expression for $i(t)$ as a fixed function of time that does not depend on what is connected at the output and term it as $i_{sc}(t)$.

$$\therefore\ i(t) = i_{sc}(t) + a_o v(t)$$
$$\text{where } i_{sc}(t) = \sum_{i=1}^{n_v} a_i v_{si}(t) + \sum_{i=1}^{n_i} b_i i_{si}(t) \quad (5.5\text{-}1)$$

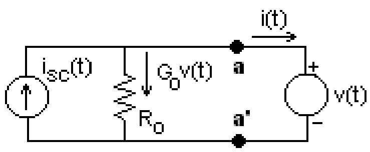

Fig. 5.5-2 A Circuit that follows Eqn. 5.5-2

This equation can be interpreted in an interesting manner if a_o can be written as $-G_o$. The number a_o can be obtained by finding out the current *delivered* to the voltage source when all the independent sources are set to zero. If the circuit contains only resistors, then, the current will actually be delivered to the circuit and hence a_o will be a negative number, making G_o a positive number. The possibility of a_o assuming a positive value does exist if there are dependent sources within the circuit. Therefore G_o is positive for a purely resistive network whereas it could be negative for a circuit containing dependent sources.

$$\therefore\ i(t) = i_{sc}(t) - G_o v(t) \quad (5.5\text{-}2)$$

This equation suggests that the current $i(t)$ is as if it is coming from an independent current source of source function $i_{sc}(t)$ that is in parallel with a resistance of $R_o = 1/G_o$. See Fig. 5.5-2.

Interpretation for $i_{sc}(t)$ in Fig. 5.5-2

How do we get the source function $i_{sc}(t)$? $i(t) = i_{sc}(t)$ when $v(t) = 0$. Therefore, we can find $i_{sc}(t)$ by finding out the current that flows out into a short-circuit that is put across its output. This is the reason why we used '*sc*' as the subscript for this current source function.

Interpretation for R_o in Fig. 5.5-2

And, how do we find out the value of R_o? If we can reduce $i_{sc}(t)$ to zero and apply a non-zero $v(t)$, the ratio of current drawn from $v(t)$ to the voltage $v(t)$ will be R_o. We can reduce $i_{sc}(t)$ to zero by deactivating all the independent sources within the circuit. Thus, we see that, R_o is nothing but the equivalent resistance of the deactivated network from terminals **a-a'**.

We conclude that a linear memoryless circuit containing resistors, dependent sources and independent sources may be replaced by a current source $i_{sc}(t)$ in parallel with a resistance R_o when it is terminated in an independent voltage source where $i_{sc}(t)$ is the current that will flow out into the short-circuit put across the terminals and R_o is the equivalent resistance of the deactivated circuit ('dead' circuit) seen from the terminals.

The circuit was terminated in an independent voltage source $v(t)$ till now. We do a mental flip now. We now choose to view that voltage source as the result of a *Substitution Operation*. That is, this voltage source came up there because we *substituted* a part of the original network by an independent voltage source by invoking *Substitution Theorem*. We note that *Substitution Theorem does not require the part of the circuit that is being substituted to be linear.* Now we bring that part of the circuit back and dispense with the independent voltage source $v(t)$. We will keep in mind that the circuit must meet all those constraints that Substitution Theorem calls for. Now we are ready to state *Norton's Theorem.*

Norton's Theorem

Statement of Norton's Theorem

Let a network with unique solution be represented as interconnection of two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 is linear and N_2 may be linear or non-linear. Then, the network N_1 may be replaced by an independent current source of value $i_{sc}(t)$ in parallel with a resistance R_o without affecting any voltage or current variable within N_2 provided the resulting network has unique solution.

$i_{sc}(t)$ is the current that will flow out into the short-circuit put across the terminals and R_o is the equivalent resistance of the deactivated circuit ('dead' circuit) seen from the terminals.

This equivalent circuit for N_1 is called its *Norton's Equivalent.* See Fig. 5.5-3.

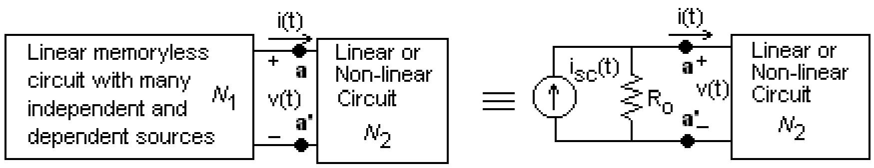

Fig. 5.5-3 Norton's Theorem and Norton's Equivalent

A similar argument after terminating the network N_1 in an independent current source of source function $i(t)$ will lead us to the conclusion that it may be replaced by an independent voltage source $v_{oc}(t)$ in series with a resistance R_o without affecting the circuit solution in N_2. $v_{oc}(t)$ in this case will be the voltage generated across **a-a'** by all the independent sources within the network N_1 when the output terminals are kept open. Therefore, it is called the open-circuit voltage. R_o will again be the equivalent resistance of the deactivated network seen from **a-a'**. The resulting equivalent circuit for N_1 is called its *Thevenin's Equivalent.*

Thevenin's Equivalent may also be derived from Norton's Equivalent by applying Source Transformation Theorem.

Thevenin's Theorem

Statement of Thevenin's Theorem

Let a network with unique solution be represented as interconnection of two networks N_1 and N_2 and let the interaction between N_1 and N_2 be only through the two terminals at which they are connected. N_1 is linear and N_2 may be linear or non-linear. Then, the network N_1 may be replaced by an independent voltage source of value $v_{oc}(t)$ in series with a resistance R_o without affecting any voltage or current variable within N_2 provided the resulting network has unique solution.

$v_{oc}(t)$ is the voltage that will appear across the terminals when they are kept open and R_o is the equivalent resistance of the deactivated circuit ('dead' circuit) seen from the terminals.

This equivalent circuit for N_1 is called its *Thevenin's Equivalent.* See Fig. 5.5-4.

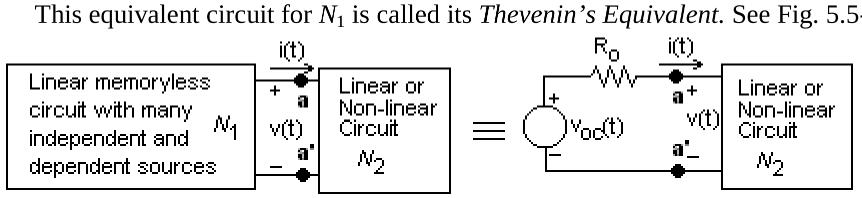

Fig. 5.5-4 Thevenin's Theorem and Thevenin's Equivalent

Compensation Theorem is a special form of Substitution Theorem for Linear Circuits. Norton's Theorem and Thevenin's Theorem are two other kinds of marriage between Substitution Theorem and Superposition Theorem. The network N_1 has to be linear since we used the idea of linear combination (*i.e.,* Superposition Theorem) in replacing it by equivalents. N_2 can be non-linear since it is subjected to only Substitution Theorem and not to Superposition Theorem.

Example : 5.5-1

Find the Thevenin's equivalent and Norton's equivalent of the circuit in Fig. 5.5-5 with respect to the terminals *a* and *b.*

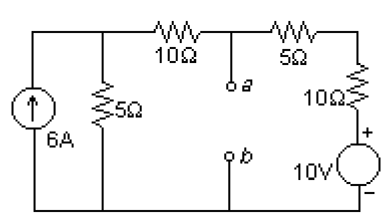

Fig. 5.5-5 Circuit for Example : 5.5-1

Solution

Step-1: Find the open-circuit voltage across *a-b*

This step may require nodal analysis or mesh analysis in general. But in simple resistive circuits like this one may use superposition principle and solve for the required voltage. The two single-source circuits needed for this are shown in Fig. 5.5-6.

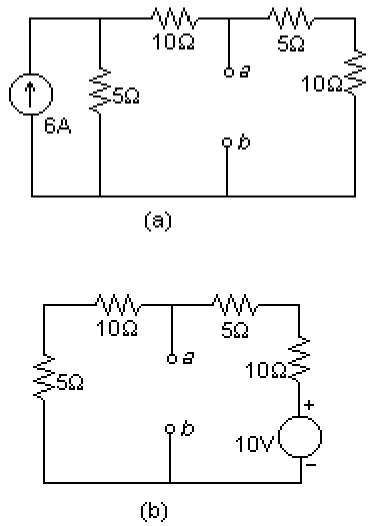

Fig. 5.5-6 Single-source Circuits for finding out contributions to open-circuit voltage across *a-b*

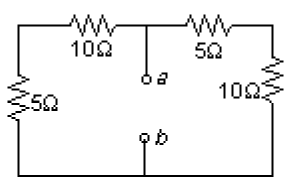

Fig. 5.5-7 The deactivated circuit for determining R_o

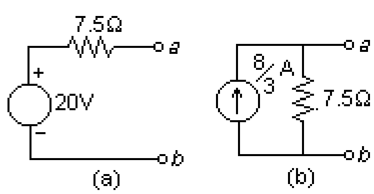

Fig. 5.5-8 (a) Thevenin's Equivalent and (b) Norton's Equivalent in Example : 5.5-1

The contribution to v_{oc} from the 6A current source = $(10+5)\times\frac{5}{5+(10+5+10)}\times 6 = 15$ volts

The contribution to v_{oc} from the 10V voltage source = $= \frac{10+5}{10+5+10+5}\times 10 = 5$ volts

Therefore, v_{oc} = 15+5 = 20 volts.

Step-2: Find Thevenin's Equivalent Resistance R_o

The deactivated circuit is shown in Fig. 5.5-7 .

The equivalent resistance seen from *a-b* = (10+5)//(10+5) = 7.5Ω

Therefore the Thevenin's equivalent and Norton's (determined by applying source transformation on Thevenin's equivalent) are as shown in Fig. 5.5-8.

Thevenin's equivalent and Norton's equivalent are equivalent only as far as the circuit variables in the network that is connected across them. They are not equivalents as far as the circuit variables in the network they replace are concerned. We agree not to seek any information on the circuit variables inside the network that was replaced by equivalent whenever we use such equivalents. For instance, *the power dissipated in R_o is not the power actually dissipated in the network that is replaced by Thevenin's equivalent.*

Determining R_o for resistive circuits is easy. Series-parallel equivalents and star-delta transformation will help us at that. But these are not useful in the case of circuits containing dependent sources since the deactivated source will still contain them. Special procedures are needed in the case of such circuits.

5.6 Determination of Equivalents for Circuits with Dependent Sources

Method-1

(i) Find v_{oc} by nodal analysis or mesh analysis or superposition principle.

(ii) Find i_{sc} by nodal analysis or mesh analysis or superposition principle.

(iii) Obtain R_o by $R_o = \frac{v_{oc}}{i_{sc}}$

Method-2

(i) Find v_{oc} by nodal analysis or mesh analysis or superposition principle.

(ii) Assume that a current source of 1 A is applied to the output terminals such that the current flows into the network at the first terminal. Carry out a node or mesh analysis and find out the voltage appearing at first terminal with respect to second terminal. The numerical value of this voltage gives the value of R_o.

(iii) Determine i_{sc} by $i_{sc} = \frac{v_{oc}}{R_o}$

Method-3

(i) Find i_{sc} by nodal analysis or mesh analysis or superposition principle.

(ii) Assume that a voltage source of 1 V is applied to the output terminals with positive polarity at the first terminal. Carry out a node or mesh analysis and find out the current flowing into that terminal. The value of this current gives the value of $G_o = 1/R_o$.

(iii) Determine v_{oc} by $v_{oc} = R_o i_{sc}$

Example : 5.6-1

Find the Thevenin's equivalent of the circuit in Fig. 5.6-1 with respect to the terminals *a* and *b* and thereby find the ratio of $v_x(t)$ to $v_S(t)$ when a resistor of 2kΩ is connected across the output. The circuit is the low-frequency signal model for a RC-coupled Common Emitter Amplifier using a bipolar junction transistor.

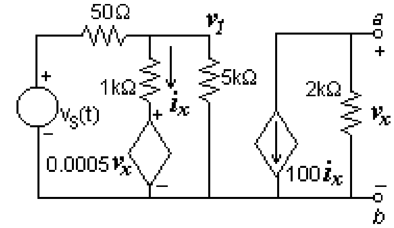

Fig. 5.6-1 Circuit for Example : 5.6-1

Solution

The first method is used in this example. Let *b* be the reference node and let the node voltage at top end of 5kΩ be v_1 as marked in the figure. Then,

$$i_x = 1\times10^{-3}(v_1 - 0.0005v_x) \text{ and } v_x = -2\times10^5 i_x = -200(v_1 - 0.0005v_x) = -200v_1 + 0.1v_x$$
$$\therefore 0.9v_x = -200v_1 \Rightarrow v_x = -222.2v_1$$
$$\therefore 0.0005v_x = -0.11v_1$$

Now writing KCL at the node where v_1 is assigned,

$$0.2\times10^{-3}v_1 + (1-(-0.11))\times1\times10^{-3}v_1 + 0.02(v_1 - v_s(t)) = 0$$
$$i.e., 0.02131v_1 = 0.02v_s(t)$$
$$\therefore v_1 = 0.9385v_s(t)$$

Since $v_x = -222.2\ v_1$, $v_x = -208.5\ v_S(t)$

Therefore $v_{oc} = -208.5\ v_S(t)$

When the terminals *a-a'* are shorted, $v_x = 0$ and therefore the independent voltage source at the input side is zero-valued. The value of v_1 under this condition is given by,

$$v_1 = \frac{5k//1k}{50+5k//1k}v_s(t) = 0.9434v_s(t)$$

Now the current i_x is $0.9434\times10^{-3}\ v_S(t)$ and hence the current in the dependent source at the output side is $0.09434v_S(t)$. All this current flows out of *a'* to *a* through the short-circuit. Hence $i_{sc}(t) = -0.09434v_S(t)$.

$$\therefore R_o = \frac{v_{oc}}{i_{sc}} = \frac{208.5v_s(t)}{0.09434v_s(t)} = 2.21k\Omega$$

The two equivalent circuits are shown in Fig. 5.6-2.

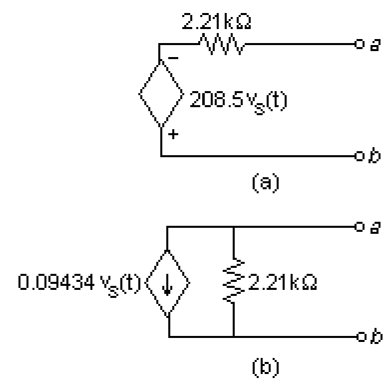

Fig. 5.6-2 (a) Thevenin's Equivalent Circuit for the Amplifier in Example:5.6-1 (b) Norton's Equivalent Circuit

If a load resistance of 2kΩ is connected at the output, the output voltage will be $-208.5v_s(t)\times\frac{2}{2+2.21} = -99.05v_s(t)$. Hence the ratio between output and input (*i.e.*, the gain of the amplifier) is – 99.05.

Example : 5.6-2

The equivalent circuit of dc current source realised using a transistor and few resistors is shown in Fig. 5.6-3. The design is expected to deliver –2mA at *a*. Find the Norton's equivalent circuit for this current source design.

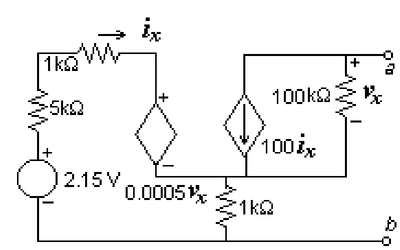

Fig. 5.6-3 Circuit for Example : 5.6-2

Solution

We find the short-circuit current at output first. The circuit for this is shown in Fig. 5.6-4.

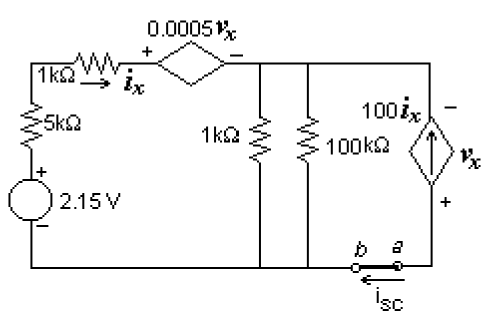

Fig. 5.6-4 Circuit for determining short-circuit current in Example : 5.6-2

We solve this circuit by mesh analysis. The current into 1k//100k (= 0.99k) is $101i_x$ .We will assume that the unit of i_x is in mA. Therefore the KVL in the first mesh is

$$-2.15 + 6\times i_x + 0.0005\times -(0.99\times 101 i_x) + 0.99\times 101 i_x = 0$$
$$i.e., 105.94 i_x = 2.15 \Rightarrow i_x = 0.0203\,\text{mA}$$

Therefore, $i_{sc} = -100\,i_x = -2.03\,\text{mA}$

To find R_o

We assume that we are injecting 1 mA into the network from terminal *a* after deactivating the circuit. We determine the voltage v_{ab}. This voltage directly gives R_o in kΩ units. Refer to circuit (a) in Fig. 5.6-5.

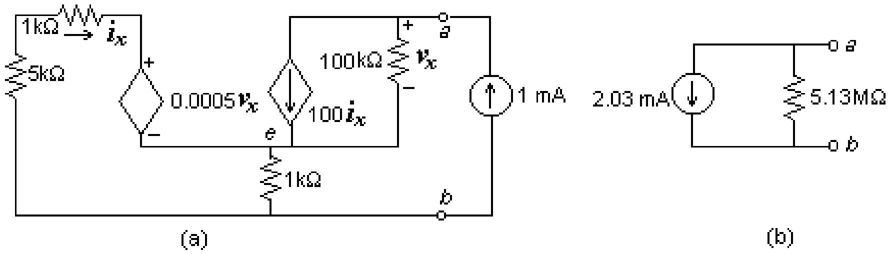

Fig. 5.6-5 (a) Circuit for Determining Thevenin's Equivalent Resistance in Example : 5.6-2
(b) The Norton's Equivalent Circuit required

Let i_x be in mA. The current in 100k resistor is $1\text{-}100i_x$ (by applying KCL at node *a*) and therefore $v_x = 100\text{-}10^4 i_x$ volts. The current into 1k resistor at node *e* is $(1+ i_x)$ mA and hence voltage of node *e* with respect to node *b* is $(1+ i_x)$ volt. Now applying KVL in the first mesh,

$$5i_x + i_x + 0.0005\times(100 - 10^4 i_x) + (1 + i_x) = 0,\ \therefore i_x = -0.5025\,\text{mA}$$
$$\therefore v_x = 100 - 10^4 i_x = 5125 \text{ volts and } \therefore v_{ab} = v_x + v_e = 5125 + (1 - 0.5025) = 5125.5\,\text{volts}$$
$$\Rightarrow R_o = 5125.5\,\text{kÙ}$$

If we really apply 1 mA into the output of the transistor circuit, either the current source we are applying will fail to function as a current source or the transistor will fail on over-voltage! But then, this is only a 'thought experiment' aimed at evaluating R_o. The Norton's equivalent circuit for this current source design is shown in Fig.5.6-5(b).

Example : 5.6-3

Find the Thevenin's equivalent of the circuit in Fig. 5.6-6 with respect to *a* and *b*.

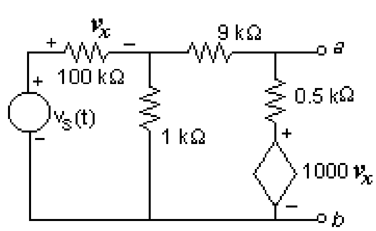

Fig. 5.6-6 Circuit for Example : 5.6-3

Solution

We find the open-circuit voltage across *a-b* first. Assume two mesh currents i_1 (mA) and i_2 (mA) in the clockwise direction in the first and second mesh respectively. The two mesh equations are

$$100i_1 + (i_1 - i_2) = v_s(t)$$
$$(i_2 - i_1) + 9i_2 + 0.5i_2 + 1000\times 100 i_1 = 0$$

Solving these two equations we get , $i_1 = \frac{v_s(t)}{9624.7}$ and $i_2 = -0.9895\ v_S(t)$.

$$\therefore v_{oc} = 0.5i_2 + 1000\times 100 i_1 = 9.9\,v_s(t)$$

To find R_o

We assume that 1 V is applied across *a-b* after deactivating the circuit and find out the current drawn by the circuit from this 1 V source. The value of current gives G_o. The circuit required to solve for R_o is shown as circuit (a) in Fig. 5.6-7.

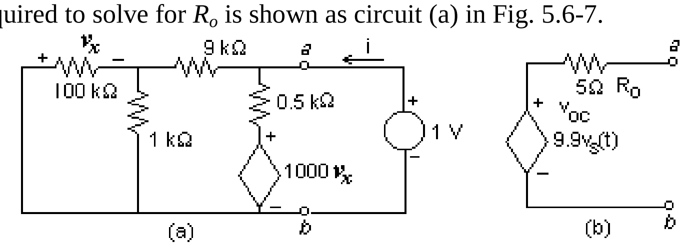

Fig. 5.6-7 (a) Circuit for Determining R_o in Example : 5.6-3 (b) The Thevenin's Equivalent

The current drawn from 1 V has two components – one flowing into the 9kΩ resistor and the second flowing into 0.5kΩ path. The first component is 1/(9+1//100) = 1/9.99 = 0.1 mA. The value of v_x is given by applying voltage division principle as -1× (100//1)/(9+100//1) = - 0.099 volts. Therefore the voltage across 0.5kΩ resistor is 1-(-0.099×1000) = 100 volts. Therefore the current flowing into 0.5kΩ path is 100/0.5 = 200 mA. Then, the total current drawn from 1 V source is 200.1 mA and the value of G_o is 0.2 S. Therefore the value of R_o is 5 Ω

The Thevenin's equivalent for the circuit is shown in (b) of Fig. 5.6-7.

5.7 Reciprocity Theorem

The nodal conductance matrix and mesh resistance matrix of a memoryless circuit without any dependent sources in it (*i.e.,* a pure resistive circuit) are symmetric matrices. Reciprocity Theorem for resistive circuits is a restatement of this fact.

Consider a pure resistive circuit with only one independent current source driving it as shown in circuit (a) of Fig. 5.7-1.

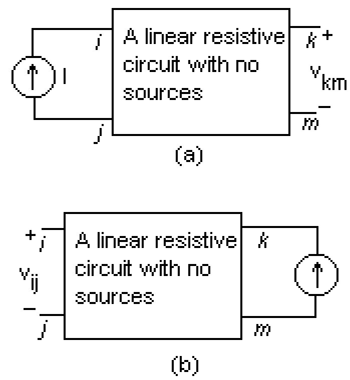

Fig. 5.7-1 Circuits to Illustrate Reciprocity Theorem

A current source of value I is connected across a node-pair i and j. Two other nodes – k and m – form a node-pair across which the voltage can be measured. There is no other independent source or dependent source inside the circuit in the box.

The nodal analysis formulation of this circuit will result in a matrix equation $\boldsymbol{YV} = \boldsymbol{I}$ where $\boldsymbol{Y}$ is a symmetric nodal conductance matrix, $\boldsymbol{V}$ is a column vector of node voltages and $\boldsymbol{I}$ is the column vector containing the net current injection at nodes. In this case $\boldsymbol{I}$ will contain I in the i^{th} row and $-I$ in the j^{th} row. All other entries will be zero. Let $\boldsymbol{A} = \boldsymbol{Y}^{-1}$. Then, $\boldsymbol{A}$ will be a symmetric matrix since $\boldsymbol{Y}$ is a symmetric matrix. We can write the node voltage vector in terms of $\boldsymbol{A}$ as $\boldsymbol{V} = \boldsymbol{AI}$. But $\boldsymbol{I}$ contains non-zero entries only in the i^{th} row and in the j^{th} row. Therefore the node voltages v_k and v_m can be written as

$$v_k = a_{ki}I - a_{kj}I$$
$$v_m = a_{mi}I - a_{mj}I$$

where a_{ki} is the element in $\boldsymbol{A}$ in k^{th} row and i^{th} column. The other a values also have same interpretation. We can now express the voltage between the two nodes as

$$v_{km} = v_k - v_m = [(a_{ki} + a_{mj}) - (a_{kj} + a_{mi})]I\ .$$

Therefore the ratio of response measured to excitation applied

$$= [(a_{ki} + a_{mj}) - (a_{kj} + a_{mi})] \qquad (5.7\text{-}1)$$

Now, consider the circuit (b) in Fig. 5.7-1. The location of excitation and response are interchanged. The current source is applied across the node-pair k and m and the voltage response is measured between the node-pair i and j.

Now the current injection vector $\boldsymbol{I}$ will have non-zero entries only in k^{th} row (= I) and in m^{th} row (= -I). Therefore we can express the node voltages at node-i and node-j as

$$v_i = a_{ik}I - a_{im}I$$
$$v_j = a_{jk}I - a_{jm}I$$

and the voltage between the two nodes as

$$v_{ij} = v_i - v_j = [(a_{ik} + a_{jm}) - (a_{jk} + a_{im})]I$$

Therefore the ratio of response measured to excitation applied

$$= [(a_{ik} + a_{jm}) - (a_{jk} + a_{im})] \qquad (5.7\text{-}2)$$

$\boldsymbol{A}$ is a symmetric matrix. Therefore $a_{ik} = a_{ki}$, $a_{mj} = a_{jm}$, $a_{jk} = a_{kj}$ and $a_{im} = a_{mi}$. Therefore, the ratios given by Eqn. 5.7-1 and Eqn. 5.7-2 are equal.

Does it matter when the two ratios in Eqn. 5.7-1 and Eqn. 5.7-2 were calculated? For instance, can we calculate the ratio in Eqn. 5.7-1 at t and the other ratio at a different instant t'? The answer, of course, is yes – provided the entries in $\boldsymbol{A}$ (*i.e.,* $\boldsymbol{Y}^{-1}$) matrix are time-invariant quantities. Hence, the circuit has to be a 'linear time-invariant resistive' one for Reciprocity Theorem to work.

First form of Reciprocity Theorem.

Statement of first form of Reciprocity Theorem

The ratio of voltage measured across a pair of terminals to the excitation current applied at another pair of terminals is invariant to an interchange of excitation terminals and response terminals in the case of a linear time-invariant resistive circuit with no independent sources inside.

The second form can be obtained by considering a dual situation shown in Fig. 5.7-2.

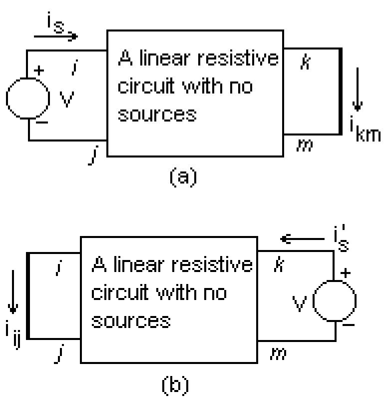

Fig. 5.7-2 Illustrating Second Form of Reciprocity Theorem

It is possible to show that the ratio $\frac{i_{km}}{V}$ is same as $\frac{i_{ij}}{V}$. It is easy to show this for a planar network using a mesh analysis formulation and exploiting the symmetry properties of mesh resistance matrix. In that case, we view i, j , k and m as mesh identifiers. One has to view the voltage source as participating in i^{th} and j^{th} meshes and the shorting link participating in k^{th} and m^{th} meshes and use an argument similar to the one we used in the case of first form of Reciprocity Theorem. However $\frac{i_{km}}{V}$ will be equal to $\frac{i_{ij}}{V}$ even for a non-planar resistive network and mesh analysis does not help us with non-planar networks.

It is possible to show that $\frac{i_{km}}{V}$ will be equal to $\frac{i_{ij}}{V}$ using nodal analysis formulation too. In that case, we view i, j , k and m as node identifiers. We view the voltage source as connected between i^{th} and j^{th} nodes and shorting link between k^{th} and m^{th} nodes in circuit (a) of Fig. 5.7-2. Then we impose the constraints that $v_i - v_j = V$ and $v_k - v_m = 0$ with a current injection of i_S at i^{th} node, $-i_S$ at j^{th} node, $-i_{km}$ at k^{th} node and i_{km} at m^{th} node. This will result in two equations in two unknowns i_{km} and i_S . We solve for i_{km}. The procedure is repeated for circuit (b) and solution for i_{ij} is obtained. Note that i_S will not be the same as i_S' . Comparison of expressions for i_{km} and i_{ij} for the same applied voltage will reveal that they are equal due to symmetry of $\boldsymbol{Y^{-1}}$ matrix.

We skip the details and state the second form of Reciprocity Theorem.

Second form of Reciprocity Theorem.

Statement of second form of Reciprocity Theorem

The ratio of current measured in a short-circuit across a pair of terminals to the excitation voltage applied at another pair of terminals is invariant to an interchange of excitation terminals and response terminals in the case of a linear time-invariant resistive circuit with no independent sources inside.

The third and last form of this theorem can be obtained by considering the circuits shown in Fig. 5.7-3.

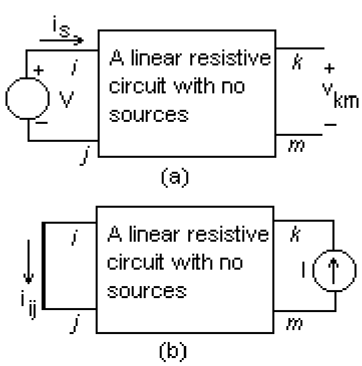

Fig. 5.7-3 Circuit Illustrating Third Form of Reciprocity Theorem

We calculate the ratio $\frac{v_{km}}{V}$ in the circuit (a) in Fig. 5.7-3 first.

Node voltage at node-i = $v_i = a_{ii}\, i_S - a_{ij}\, i_S$
Node voltage at node-j = $v_j = -a_{jj}\, i_S + a_{ji}\, i_S$
Node voltage at node-k = $v_k = a_{ki}\, i_S - a_{kj}\, i_S$
Node voltage at node-m = $v_m = a_{mi}\, i_S - a_{mj}\, i_S$
The node voltages at node-i and node-j are constrained to have a difference of V.

$$\therefore \quad a_{ii}\, i_S - a_{ij}\, i_S - (-a_{jj}\, i_S + a_{ji}\, i_S) = V \Rightarrow i_s = \frac{V}{a_{ii} + a_{jj} - a_{ij} - a_{ji}}$$

Substituting this expression for i_S in the equations for v_k and v_m we get the ratio of voltage measured across the second pair of terminals to the voltage applied at the first pair of terminals as

$$\frac{v_{km}}{V} = \frac{(a_{ki} - a_{mi}) + (a_{mj} - a_{kj})}{a_{ii} + a_{jj} - a_{ij} - a_{ji}} \qquad (5.7\text{-}3)$$

Now we calculate $\frac{i_{ij}}{I}$ in the circuit (b) in Fig. 5.7-3.

$v_i = a_{ii}\, i_{ij} - a_{ij}\, i_{ij} + a_{ik}\, I - a_{im}\, I$

$v_j = -\, a_{jj}\, i_{ij} + a_{ji}\, i_{ij} + a_{jk}\, I - a_{jm}\, I$

But $v_i = v_j$

$\therefore\ 0 = (a_{ii}\, i_{ij} - a_{ij}\, i_{ij} + a_{ik}\, I - a_{im}\, I\,) - (-\, a_{jj}\, i_{ij} + a_{ji}\, i_{ij} + a_{jk}\, I - a_{jm}\, I)$

Solving this we get the ratio of current measured in short-circuit across the first pair of terminals to the current source applied at across the second pair of terminals as

$$\frac{i}{I} = \frac{(a_{ik} - a_{im}) + (a_{jm} - a_{jk})}{a_{ii} + a_{jj} - a_{ij} - a_{ji}} \qquad (5.7\text{-}4)$$

Comparing the two expressions in Eqn. 5.7-3 and Eqn. 5.7-4 and using the symmetry of ***A*** (*i.e.*, the inverse of nodal conductance matrix) we see that the two ratios are equal.

<u>Third form of Reciprocity Theorem.</u>

The ratio of current measured in a short-circuit across first pair of terminals to the excitation current applied at the second pair of terminals is same as the ratio of voltage measured across the second pair of terminals to the voltage applied at the first pair of terminals in the case of a linear time-invariant resistive circuit with no independent sources inside. (Refer Fig. 5.7-3 for polarity of currents and voltages)

Statement of third form of Reciprocity Theorem.

Reciprocity Theorem is not used in routine circuit analysis as frequently as Superposition Theorem and Thevenin's and Norton's Theorems are. However, it comes in handy in the analysis of two-port networks. It helps to ease measurement issues in circuits sometimes. It is valid for circuits containing linear inductor, capacitors and mutual inductors also. We will prove that when we take up dynamic circuits for detailed study in later chapters.

Note that the key to Reciprocity Theorem is that (i) the nodal conductance matrix ***Y*** (and mesh resistance matrix ***Z***) of the circuit must be time-invariant and *symmetric* and (ii) excitation should be applied only at terminals identified, *i.e.,* there should not be independent sources present within the circuit. ***Y*** and ***Z*** matrices of a circuit containing *linear two-terminal time-invariant resistors* will be symmetric. Hence such circuits will obey all the three forms of Reciprocity Theorem unconditionally.

Dependent sources, even if they are linear, bilateral and time-invariant, *can* make these matrices asymmetric. *But they need not do so always*. There can be dependent sources in the circuit and yet the circuit may have symmetric ***Y*** and ***Z*** matrices. Reciprocity Theorem will hold for such circuits too. See the following example.

Example : 5.7-1

Show that Reciprocity Theorem is valid for the circuit in Fig. 5.7-4.

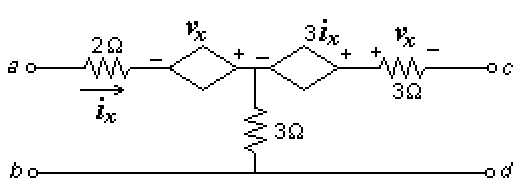

Fig. 5.7-4 Circuit for Example : 5.7-1

Solution

We find the mesh resistance matrix of the circuit first and verify whether it is symmetric and time-invariant. We know that the mesh resistance matrix of a circuit can be found from its deactivated version. Since excitation can be applied only across *a-b* and *c-d* we short these two ports (since for mesh analysis voltage source is the excitation source) and get the circuit in Fig. 5.7-5. The mesh currents are identified in it.

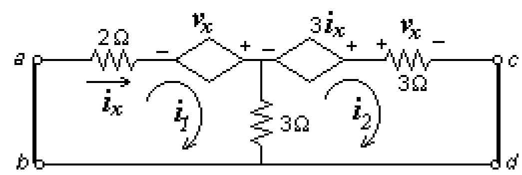

Fig. 5.7-5 Circuit to Obtain ***Z*** Matrix in Example : 5.7-1

The mesh equations are written for the two meshes after observing that $i_x = i_1$ and $v_x = 3i_2$.

$2\,i_1 - 3\,i_2 + 3\,(i_1 - i_2) = 0$ and $3\,(i_2 - i_1) - 3\,i_1 + 3\,i_2 = 0$ are the mesh equations. Therefore the mesh resistance matrix ***Z*** is,

$\boldsymbol{Z} = \begin{bmatrix} 5 & -6 \\ -6 & 6 \end{bmatrix}$ and it is a symmetric time-invariant matrix. Therefore Reciprocity theorem will be valid in the circuit.

We verify the first form by using the circuit configurations shown in Eqn. 5.7-4. A 1 A independent current source is used to drive the *a*-*b* terminal pair first and the voltage v_{cd} is noted. Then the same current source is used to drive the *c*-*d* terminal pair and the voltage v_{ab} is noted. We expect to see that $v_{ab} = v_{cd}$.

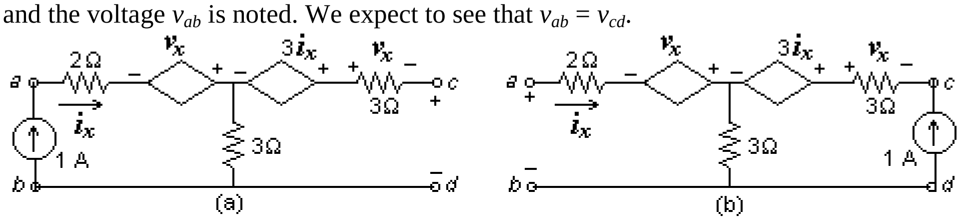

Fig. 5.7-6 Circuits for Verifying Reciprocity Theorem in Example : 5.7-1

The second mesh current in circuit (a) is zero. First mesh current is 1 A. Applying KVL in the second mesh, $-3\times1 - 3\times1 + 3\times0 + v_{cd} = 0 \Rightarrow v_{cd} = 6$ volts.

The first mesh current in circuit (b) is zero. Second mesh current is -1 A. Applying KVL in the first mesh, $-v_{ab} + 2\times0 - (3\times-1) + 3\times1 = 0 \Rightarrow v_{ab} = 6$ volts.

Thus we see that first form of the theorem holds in this circuit. It may be verified in a similar manner that the other two forms are also valid in this circuit.

5.8 Maximum Power Transfer Theorem

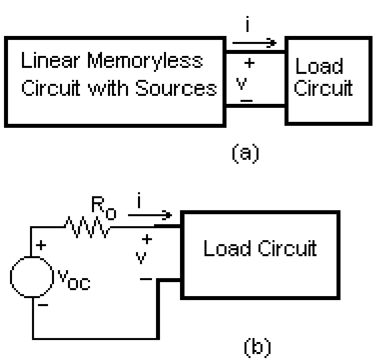

Fig. 5.8-1 (a) The Power Delivery Context (b) Power Delivery Circuit Replaced by its Thevenin's Equivalent

All electrical and electronic circuits fall under one of the three broad categories – power generation and delivery circuits, power conditioning circuits and signal generation and conditioning circuits.

In a power delivery context, one part of the circuit acts as a power source and delivers power to the other part of the circuit. In the process of delivering power to load part of the circuit, the source part of the circuit ends up dissipating some of the power within itself. This compromises the efficiency of power delivery as well as the power availability to the load at the same time. Hence the power delivery capability of source part of the circuit for a given load circuit is of crucial practical significance – both in high-power electrical circuits (kW to 100's of MW) and low-power electronic circuits (pW to 100's of W). We address the issue of power delivery capability of a source circuit in this section.

Fig. 5.8-1 shows a linear time-invariant memoryless circuit containing one or more independent dc sources delivering power to a load circuit which may be linear or non-linear. It is assumed that the constraints required for applying Thevenin's theorem are satisfied by the entire circuit – that is, the circuits in (a) and (b) have unique solution and there is no interaction between the delivery circuit and load circuit other than through the common terminals. Then, we can replace the power delivery circuit by its Thevenin's equivalent comprising an open-circuit voltage in series with the Thevenin's equivalent resistance.