angular frequency of the applied sinusoidal input. Carrying out this operation in Eqn. 12.2-1, we get the *frequency response function* $H(j\omega)$ as

$$H(j\omega) = \frac{\sum_{k=0}^{m} b_k (j\omega)^k}{(j\omega)^n + \sum_{i=0}^{n-1} a_i (j\omega)^i}$$

If $x = A\cos\omega t$, then, $y = |H(j\omega)|\, A\cos(\omega t + \phi)$ where $\phi = \angle H(j\omega)$.

12.3 Time-Domain Convolution Integral

We show in this section that the impulse response is a complete characterization of a linear time-invariant circuit. It is a complete characterization in the sense that the zero-state response of a linear time-invariant circuit to application of any arbitrary input source function can be determined from impulse response and input source function.

We show this in three steps. In the first step we show that the zero-state response of a linear time-invariant circuit to a narrow rectangular pulse input of unit area content can be approximated by its impulse response. In the second step we show that any arbitrary input source function can be resolved into a stream of weighted impulses. In the third step we show how the zero-state response of a linear time-invariant circuit to an arbitrary input source function at any time instant t can be constructed by superposing weighted and time-shifted impulse responses corresponding to impulse-stream resolution of the input source function arrived at in second step.

Zero-State Response to Narrow Rectangular Pulse Input

Let $x(t)$ be the input source function and $y(t)$ be the response function in a n^{th} order linear time-invariant circuit. We identify two special response functions. They are unit step response and unit impulse response. Let $s(t)$ represent the unit step response and $h(t)$ represent the unit impulse response of the system. Note that these two are only special symbols for $y(t)$ when $x(t) = u(t)$ and $x(t) = \delta(t)$ respectively.

Now we let $x(t)$ be a narrow rectangular pulse with duration of $\Delta\tau$ sec and height of $1/\Delta\tau$ so that it contains unit area under the waveform. It is centered about $t = 0$. Thus $x(t)$ is given by

$$x(t) = \begin{cases} 0 \ \text{ for } \ -\infty < t < -\dfrac{\Delta\tau}{2} \\ \dfrac{1}{\Delta\tau} \ \text{ for } \ -\dfrac{\Delta\tau}{2} < t < \dfrac{\Delta\tau}{2} \\ 0 \ \text{ for } \ \dfrac{\Delta\tau}{2} < t < \infty \end{cases}$$

This waveform can be expressed as sum of two scaled step functions as below.

$$x(t) = \frac{1}{\Delta\tau}\left[u(t + \tfrac{\Delta\tau}{2}) - u(t - \tfrac{\Delta\tau}{2})\right]$$

Therefore, the zero-state response of the circuit to $x(t)$ can be obtained as the sum of zero-state response to $\frac{1}{\Delta\tau}u(t + \frac{\Delta\tau}{2})$ and zero-state response to $-\frac{1}{\Delta\tau}u(t - \frac{\Delta\tau}{2})$. The zero-state response to $u(t)$ is the step response $s(t)$. The circuit is a *time-invariant* one. Hence when the input source function is advanced by $\Delta\tau/2$ sec, the response also gets advanced by the same amount in time-axis. Similarly, when the input source function is delayed by $\Delta\tau/2$ sec, the response also gets delayed by the same amount in time-axis. Therefore, the zero-state response to $\frac{1}{\Delta\tau}u(t + \frac{\Delta\tau}{2})$ is $\frac{1}{\Delta\tau}s(t + \frac{\Delta\tau}{2})$ and the zero-state response to $-\frac{1}{\Delta\tau}u(t - \frac{\Delta\tau}{2})$ is $-\frac{1}{\Delta\tau}s(t - \frac{\Delta\tau}{2})$. We have employed the fact the circuit is a linear one to scale the response by $\frac{1}{\Delta\tau}$. Therefore, the zero-state response to narrow rectangular pulse input of unit area can be expressed as $y(t) = \frac{1}{\Delta\tau}\left[s(t+\frac{\Delta\tau}{2}) - s(t-\frac{\Delta\tau}{2})\right]$.

Now we let $\Delta\tau$ approach zero. That is, we reduce the width of the pulse and increase its height such that the area under the waveform remains unity. Then, the response can be expressed as $y(t) = \lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau}\left[s(t+\frac{\Delta\tau}{2}) - s(t-\frac{\Delta\tau}{2})\right]$. But this limit is nothing but the first derivative of $s(t)$ evaluated at t. Therefore, $y(t) = \lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau}\left[s(t+\frac{\Delta\tau}{2}) - s(t-\frac{\Delta\tau}{2})\right] = \frac{ds(t)}{dt}$. But derivative of step response of a linear time-invariant circuit is nothing but its impulse response since $\delta(t)$ is the first derivative of $u(t)$ (in the sense that integrating $\delta(t)$ yields $u(t)$). Therefore, response for a narrow rectangular pulse of unit height approaches impulse response in a linear time-invariant circuit when the width of the pulse is made arbitrarily small, keeping the area at unity. Hence, the response for a sufficiently narrow pulse of area A can be approximated by A times unit impulse response. The approximation will get better when the width of pulse reaches very low values compared to time constants in the circuit.

Expansion of an Arbitrary Input Function in Terms of Impulse Functions

Now let $x(\tau)$ be an arbitrary input source function which is assumed to be known in the entire time-axis. We denote the time-variable by τ this time for a reason that will emerge soon.

We divide the entire τ-axis into small strips (*i.e.,* intervals) of width $\Delta\tau$ as shown in Fig. 12.3-1 .

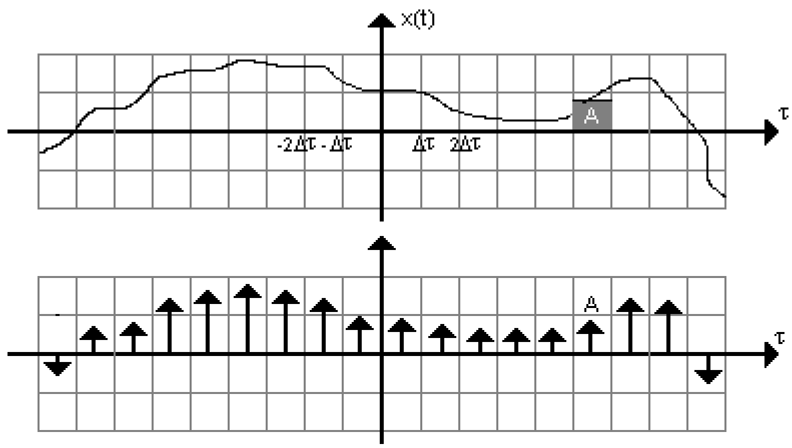

Fig. 12.3-1 An Arbitrary Input Function Approximated by an Impulse Stream

$x(\tau)$ varies within each strip. However, any curve can be treated as a straight-line if the interval of observation is sufficiently small. Therefore, we may approximate the waveform by straight-line segments within small intervals of width $\Delta\tau$. Then the area under the waveform for each waveform-strip is given by the value of $x(\tau)$ at the middle of the interval multiplied by the width of the strip. This area content can be expressed as $x(n\Delta\tau\text{-}\Delta\tau/2) \times \Delta\tau$. Now we replace this strip by a rectangular strip with same area. Obviously, all these approximations will tend to become exact as $\Delta\tau \to 0$. We have shown in the previous sub-section that applying a rectangular pulse of area A to a linear time-invariant circuit results in a response that can be approximated by A times the impulse response of the circuit provided the width of the rectangular pulse is very small. Therefore, each waveform strip under intervals of $\Delta\tau$ can be replaced by impulses of magnitude equal to the area under the waveform strip as far as the effect of $x(\tau)$on a linear time-invariant circuit is concerned. The impulses are to be located at the center of the corresponding intervals. This is shown in Fig. 12.3-1. Let us denote the resulting impulse stream by $x_i(\tau)$. Then $x_i(\tau)$ can be expressed as

$$x_i(\tau) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} x[(n-\tfrac{1}{2})\Delta\tau]\delta[\tau-(n-\tfrac{1}{2})\Delta\tau]\Delta\tau \qquad (12.3\text{-}1)$$

An arbitrary input function $x(t)$ expressed as a sum of scaled and shifted impulses – called 'impulse resolution of $x(t)$'

The 'n' in Eqn. 12.3-1 is an integer. This equation expresses an arbitrary function $x(\tau)$ as a sum of infinitely many scaled and time-shifted impulse functions. The response of a linear time-invariant circuit to $x(\tau)$ is approximately the same as response of the same circuit to $x_i(\tau)$ as given by Eqn. 12.3-1. The approximation becomes exact as $\Delta\tau \to 0$.

The Convolution Integral

Convolution Integral expresses the zero-state response of a linear time-invariant circuit in terms of its impulse response and the input source function.

Only the properties of 'linearity' and 'time-invariance' are employed in the derivation of this integral. Hence Convolution Integral is only a restatement of properties of 'linearity' and 'time-invariance'

Now we address the issue of determining the response of a linear time-invariant circuit from its impulse response and the input source function.

We consider the most general case of a two-sided input source function and two-sided impulse response. The impulse response of a physical linear time-invariant circuit will invariably be one-sided, *i.e.*, $h(t) = 0$ for $t < 0$. This is so since all physical systems are *causal systems*. A two-sided impulse response implies that the system could somehow predict that an impulse is going to hit at $t = 0$ and it started responding in anticipation from infinite past onwards. This clearly goes against the law of causality. However, since we impose only two requirements on the system now – that it be linear and time-invariant – and do not impose the requirement of *causality* at this point in our discussion, we use a two-sided $h(t)$ in this discussion. Moreover, we assume that we have full knowledge of the input applied to the circuit from $t = -\infty$ to $+\infty$.

Let us imagine that we want to find out y at a time-instant t located on the τ-axis.

The input $x(\tau)$ can be treated as a stream of scaled and shifted impulses as explained in the previous sub-section. Each impulse in the impulse stream starts a new scaled-impulse response in the circuit. Since the circuit is time-invariant, the response to a shifted impulse is same as the impulse response itself shifted by the same time-shift. That is, response to $\delta[\tau-(n-\tfrac{1}{2})\Delta\tau]$ is equal to $h[\tau-(n-\tfrac{1}{2})\Delta\tau]$. The impulse $\delta[\tau-(n-\tfrac{1}{2})\Delta\tau]$ is located at the middle of n^{th} interval in τ-axis and it is scaled by the value of x at the middle of that interval$\times\Delta\tau$. The impulse response due to this single impulse will contribute to the value of output at t. This contribution can be obtained as the instantaneous value of the scaled and shifted impulse response given by $x[(n-\tfrac{1}{2})\Delta\tau]\delta[\tau-(n-\tfrac{1}{2})\Delta\tau]\Delta\tau$ evaluated at $\tau = t$. Therefore,

Contribution from the impulse located at $\tau = (n-\tfrac{1}{2})\Delta\tau$ to the output y at t = $x[(n-\tfrac{1}{2})\Delta\tau]\delta[\tau-(n-\tfrac{1}{2})\Delta\tau]\Delta\tau \Big|_{\tau=t} = x[(n-\tfrac{1}{2})\Delta\tau]\delta[t-(n-\tfrac{1}{2})\Delta\tau]\Delta\tau$.

If the system is causal, its impulse response will be right-sided and hence only those impulses that appear *before or at t* in τ-axis will contribute to the output at t. But we are considering a two-sided impulse response here. Therefore, even those impulses which are yet to appear will contribute to the output at t. That is, all impulses from $-\infty$ to $+\infty$ in τ-axis will contribute to y at t. The circuit is stated to be linear and it obeys superposition principle. Thus, the *instantaneous* value of y at t can be constructed by adding all the contributions to output from all the impulses in Eqn. 12.3-1.

$$\therefore y(t) \approx \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} x[(n-\tfrac{1}{2})\Delta\tau]\delta[t-(n-\tfrac{1}{2})\Delta\tau]\Delta\tau \qquad (12.3\text{-}2)$$

The result in Eqn. 12.3-2 is only approximate since Eqn. 12.3-1 is only an approximation. We take the limit of Eqn. 12.3-2 as $\Delta\tau \to 0$ to make the result exact.

$$\therefore y(t) = \lim_{\Delta\tau \to 0} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} x[(n-\tfrac{1}{2})\Delta\tau]\delta[t-(n-\tfrac{1}{2})\Delta\tau]\Delta\tau$$

The variable $(n-\frac{1}{2})\Delta\tau$ is a discrete variable. It represents a sequence of equidistant time-points in the τ-axis. The spacing between these equidistant time-points go to zero as $\Delta\tau\to 0$ and in the limit they cover every point in τ-axis. Equivalently, the variable $(n-\frac{1}{2})\Delta\tau$ gets replaced by the variable τ. Moreover, the summation becomes integration as $\Delta\tau\to 0$. Therefore,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \tag{12.3-3}$$

The integral in Eqn. 12.3-3 is called the *Convolution Integral* and the process of evaluating this integral is called '*convolving* $x(t)$ *with* $h(t)$'. The convolution operation is also indicated by $x(t)*h(t)$. Therefore,

Convolution Integral for total response of a circuit with a two-sided impulse response $h(t)$ excited by a two-sided input function $x(t)$

$$y(t) = x(t)*h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Any two time-functions can be convolved. When the first one is an input to a linear time-invariant circuit and the second one is the impulse response of that circuit, the convolution between them yields the response of the circuit.

Note that the integral is a definite integral yielding a single value that denotes the instantaneous value of y at a particular time instant t. Therefore, determination of one value in the output waveform requires the evaluation of an integral over the entire time-axis.

But which response does the Eqn. 12.3-3 give us? Total response, forced response or zero-state response?

We had assumed that the input source function $x(t)$ is known from -∞ to ∞ in the time-axis. Hence the output given by Eqn. 12.3-3 is the forced response which also happens to be the total response since there is no transient response when the input starts from -∞.

But, in practice, the input function is known only from $\tau = 0$. Then we assume that $x(\tau) = 0$ for $\tau < 0$. The response obtained with that assumption is what we called zero-state response. Therefore, the response given by convolution integral is the zero-state response if input source function is one-sided. Convolution integral can not yield the zero-input response part. That has to be found out separately. Also, the limit of integration can now be changed to 0 to ∞. Therefore,

Convolution Integral for zero-state response of a circuit with a two-sided impulse response $h(t)$ excited by a right-sided input function $x(t)$

$$y_{zsr}(t) = x(t)*h(t) = \int_{0}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \;\text{ if }\; x(t)=0 \;\text{ for } t<0 \tag{12.3-4}$$

If the impulse response is causal, $h(t)$ will be right-sided. Therefore the value of integrand in Eqn. 12.3-3 will be zero for all $\tau > t$. Therefore the upper limit on the integral can be set at t. Therefore,

Convolution Integral for total response of a circuit with a causal impulse response $h(t)$ excited by a two-sided input function $x(t)$

$$y(t) = x(t)*h(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau \;\text{ if }\; h(t)=0 \;\text{ for } t<0 \tag{12.3-5}$$

If both $x(t)$ and $h(t)$ are right-sided, convolution integral returns the zero-state response of a causal linear time-invariant circuit and the integration limits can be set at 0 and t.

Convolution Integral for zero-state response of a circuit with a causal impulse response $h(t)$ excited by a right-sided input function $x(t)$

$$y_{zsr}(t) = x(t)*h(t) = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau \;\text{ if }\; x(t) \text{ and } h(t)=0 \;\text{ for } t<0 \tag{12.3-6}$$

Example : 12.3-1

The impulse response of a linear time-invariant circuit is $h(t) = 2e^{-t}\,u(t)$ and the input to the circuit is $x(t) = e^{-2t}\,u(t)$. The circuit was initially relaxed. Find its output by the convolution $x(t) * h(t)$. Also find the convolution $h(t)*x(t)$ and verify that it is the same as $x(t) * h(t)$.

Solution

Since the circuit was initially relaxed, the total response is given by the zero-state response itself. And, zero-state response to any input is obtained by convolving the input source function with the impulse response. Therefore the output $y(t)$ is obtained as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_0^t e^{-2\tau}.2e^{-(t-\tau)}\, d\tau$$

The limits of integration were changed from -∞ to 0 and ∞ to t since both the input source function and impulse response are right-sided functions of time.

$$y(t) = \int_0^t e^{-2\tau}.2e^{-(t-\tau)}\, d\tau = 2\int_0^t e^{-2\tau} e^{\tau} e^{-t}\, d\tau = 2\int_0^t e^{-\tau} e^{-t}\, d\tau$$

The integration variable is τ, and not t. Therefore the factor e^{-t} can be pulled out of integration sign as below.

$$y(t) = 2e^{-t}\int_0^t e^{-\tau}\, d\tau = 2e^{-t}[-e^{-\tau}]\Big|_0^t = 2e^{-t}[1-e^{-t}] = 2e^{-t} - 2e^{-2t}$$

Any two time-functions can be convolved with each other. Hence we can convolve $h(t)$ with $x(t)$. Then,

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_0^t 2e^{-\tau}.e^{-2(t-\tau)}\, d\tau = 2e^{-2t}\int_0^t e^{\tau}\, d\tau = 2e^{-2t}[e^{t} - 1] = 2e^{-t} - 2e^{-2t}$$

We see that $x(t)*h(t) = h(t)* x(t)$ in this example. But, $h(t)* x(t)$ can be interpreted as the output of a linear time-invariant circuit with $x(t)$ as its impulse response and $h(t)$ as its input. Thus, the roles of impulse response and input source function seem to be interchangeable as far as this example is concerned. In fact it is generally true as shown below. We use a substitution of variable $\tau' = t - \tau$. Then $\tau = t - \tau', d\tau = -d\tau'$ and

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = -\int_{\infty}^{-\infty} h(t-\tau')\, x(\tau')\, d\tau' = \int_{-\infty}^{\infty} x(\tau')\, h(t-\tau')\, d\tau' = x(t) * h(t)$$

If $x_1(t)$ and $x_2(t)$ are two arbitrary functions of time, then $x_1(t) * x_2(t) = x_2(t) * x_1(t)$

Therefore, the role of impulse response and input source function are interchangeable in a linear time-invariant circuit.

Example : 12.3-2

The impulse response of a second order linear time-invariant circuit is seen to be $h(t) = 2e^{-t}\cos 2t\, u(t)$. Find its steady-state output when it is driven by $x(t) = \cos 4t\; u(t)$ by convolution.

Solution

We find the zero-state response to $x(t) = \cos 4t$ first.

$$h(t) = 2e^{-t}\cos 2t\, u(t) = 2e^{-t}\,\frac{e^{j2t} + e^{-j2t}}{2}\,u(t) = \left[e^{(-1+j2)t} + e^{(-1-j2)t}\right]u(t)$$

$$x(t) = 2\cos t\, u(t) = \left[e^{jt} + e^{-jt}\right]u(t)$$

We have used Euler's formula in these steps.

$$y(t)=x(t)*h(t)=\int_0^t (e^{j\tau}+e^{-j\tau})[e^{(-1+j2)(t-\tau)}+e^{(-1-j2)(t-\tau)}]d\tau$$
$$=\int_0^t (e^{j\tau}+e^{-j\tau})[e^{(-1+j2)(t-\tau)}]+\int_0^t (e^{j\tau}+e^{-j\tau})[e^{(-1-j2)(t-\tau)}]d\tau$$
$$=e^{(-1+j2)t}\int_0^t (e^{j\tau}+e^{-j\tau})[e^{-(-1+j2)\tau}]d\tau+e^{(-1-j2)t}\int_0^t (e^{j\tau}+e^{-j\tau})[e^{-(-1-j2)\tau}]d\tau$$
$$=e^{(-1+j2)t}\int_0^t [e^{(1-j)\tau}+e^{(1-j3)\tau}]d\tau+e^{(-1-j2)t}\int_0^t [e^{(1+j3)\tau}+e^{(1+j)\tau}]d\tau$$
$$=e^{(-1+j2)t}\left[\frac{e^{(1-j)t}-1}{1-j}+\frac{e^{(1-j3)t}-1}{1-j3}\right]+e^{(-1-j2)t}\left[\frac{e^{(1+j)t}-1}{1+j}+\frac{e^{(1+j3)t}-1}{1+j3}\right]$$
$$=\left[\frac{e^{(j)t}-e^{(-1+j2)t}}{1-j}+\frac{e^{(-j)t}-e^{(-1+j2)t}}{1-j3}\right]+\left[\frac{e^{(-j)t}-e^{(-1-j2)t}}{1+j}+\frac{e^{(j)t}-e^{(-1-j2)t}}{1+j3}\right]$$
$$=\left[\frac{e^{(j)t}}{1-j}+\frac{e^{(-j)t}}{1+j}+\frac{-e^{(-1+j2)t}}{1-j}+\frac{-e^{(-1-j2)t}}{1+j}+\right.$$
$$\left.\frac{e^{(-j)t}}{1-j3}+\frac{e^{(j)t}}{1+j3}+\frac{-e^{(-1+j2)t}}{1-j3}+\frac{-e^{(-1-j2)t}}{1+j3}\right]$$
$$=\left[\cos t-\sin t-e^{-t}\cos 2t+e^{-t}\sin 2t+\frac{1}{5}\cos t+\frac{3}{5}\sin t-\frac{1}{5}e^{-t}\cos t-\frac{3}{5}e^{-t}\sin t\right]$$

Note that the evaluation of convolution integral in this example has been rendered easy by the liberal use of Euler's formula. Integrating exponential functions is an easy job. We do not need table of integrals or integration by parts for that.

The sinusoidal steady-state component in this output is = cos$t$-sin$t$ +0.2cos$t$ +0.6sin$t$= 1.2cos$t$ – 0.4sin$t$ = 1.265cos(t + 18.44^0).

Steady-state response to an input function (provided the notion of steady-state is applicable with that particular input) can be found in two ways. In the first method we find out the zero-state response when the particular input is applied to the circuit and accept the part of zero-state response that remains after the natural response terms damp down to zero as the steady-state response. This is the approach we followed in this example.

This does not imply that the steady-state response can be found as the limit of zero-state response as $t \to \infty$. That is a wrong interpretation of 'steady-state'. Steady-state response is the response that remains after the transient terms die down. The response that remains after transients die down need not be a *constant* value. Therefore a *limit* operation is not valid. It may appear to be valid in the case of dc steady-state. However, viewing steady-state response as the limit of response as $t \to \infty$ is *not a correct interpretation*.

The reader is cautioned against a common error in interpreting 'steady-state' !

The second approach is to assume that the input was applied to the circuit from t = -∞ onwards. In that case we would be determining the forced response of the circuit and forced response is the steady-state response whenever the notion of steady-state is applicable. Therefore, the convolution integral $y(t)=\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)d\tau$ will yield the steady-state response straightaway if $x(t)$ is assumed to have been applied from t = -∞ onwards. Of course, the integral specializes to $y(t)=\int_{-\infty}^{t} x(\tau)\,h(t-\tau)d\tau$ for a causal circuit. The reader may verify the steady-state result by this method in the case of this example.

Example : 12.3-3

Show that the steady-state step response value is equal to area under the impulse response for a stable linear time-invariant circuit.

Solution

We use the second approach mentioned under the previous example to determine the steady-state. We assume that unit input was applied to the circuit from t = -∞ onwards. Then,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau \;(\because \text{Role of } x(t) \text{ and } h(t) \text{ can be interchanged})$$

$$= \int_{-\infty}^{\infty} h(\tau)\,d\tau \;(\because \text{input is unit constant})$$

$$= \text{Area under Impulse Response}$$

Therefore forced response to unit dc input is equal to area under impulse response. But forced response to unit dc input is same as steady-state response to unit step input. Therefore, steady-state step response value is equal to area under impulse response.

Graphical Interpretation of Convolution in Time-Domain

We develop a graphical interpretation for convolution integral $y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$ in this sub-section and thereby appreciate why the impulse response function is often called the *Scanning Function*.

Two functions are involved in a product in the integrand. The first one is simple – it is x laid out in τ-axis. The second one, $h(t-\tau)$, calls for some interpretation before we can visualize it.

$h(\tau)$ is the circuit impulse response laid out in τ-axis. We define a new function $h_i(\tau)$as $h_i(\tau) = h(-\tau)$. For every τ, the value of h_i is the value that h has at $-\tau$. This means that the graph of $h_i(\tau)$ will be the mirror image of the graph of $h(\tau)$ with the mirror image taken about the vertical axis. For instance, if $h(\tau)$ is a right-sided function, then, $h_i(\tau)$ will be a left-sided function.

Now we construct a delayed version of $h_i(\tau)$ and call it $h_{id}(\tau)$. The delay involved is taken as t sec. Thus $h_{id}(\tau) = h_i(\tau-t)$. But since $h_i(\tau)$ is the mirror image of $h(\tau)$, $h_i(\tau-t)$ must be equal to $h[-(\tau-t)] = h(t-\tau)$. Therefore, $h(t-\tau)$ is a waveform obtained by mirror-reflecting $h(\tau)$ about vertical axis and moving it forward in τ-axis by t sec if t is positive or moving it backward by $|t|$ sec if t is negative. t is the value of time-instant at which the output y is to be calculated. The two waveforms $x(\tau)$ and $h(t-\tau)$ are multiplied to form the product waveform and the area under the product waveform is obtained in order to determine *one value* of y at a particular time-instant t. The process is repeated for different values of t to construct y as a function of time.

Fig. 12.3-2 shows convolution graphically for $x(t) = e^{-0.5\,t}[u(t)-u(t-5)]$ and $h(t) = e^{-t}u(t)$ for four values of t. The shaded curves show the integrand and shaded area gives the output values.

The impulse response function moves in from the left side of time-axis and sweeps through the input source function as t increases. The overlap region between the mirror-reflected and shifted impulse response and the input source function increases with t. Impulse response function does a kind of *selective scanning-in* on the input function as t increases. It places more emphasis on the recent values of input; nevertheless it *scans in* all the past values of input too – but scales them down depending on the time interval between current instant and the past instant.

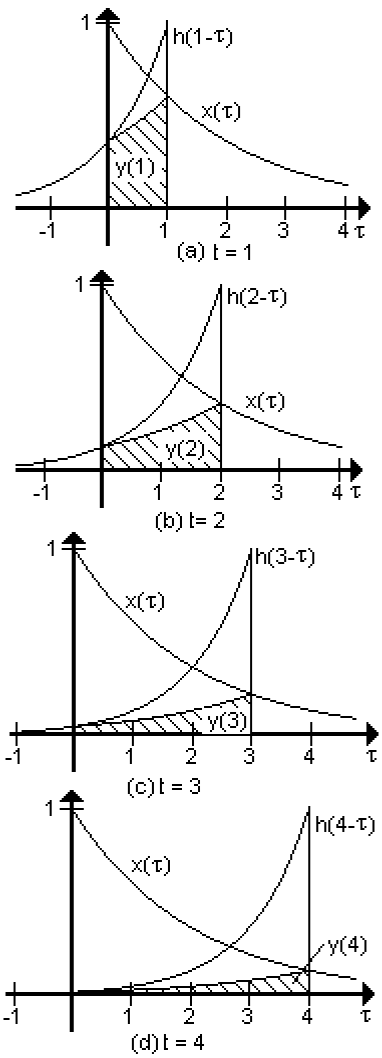

Fig. 12.3-2 Graphical Interpretation of Convolution

The impulse response of a stable linear time-invariant circuit can contain only decreasing exponential functions and hence it will be a function that tapers down to zero as time increases. Therefore, though the impulse response scans in the past input values from infinite past theoretically, the significance of scanned-in past values will be negligible after a certain point in the past. This is due to the tapering nature of impulse response. For instance, an exponential impulse response function goes down to 1% of its initial value in about 5 time constants. Thus we may assume without much error that input values that occurred before 5 time constants of impulse response will not influence the output significantly. This may equivalently be stated as 'the depth of memory of the circuit is ~ 5 time constants of impulse response'. If impulse response contains many exponential functions, the largest time constant in impulse response is to be used to decide the depth of circuit memory.

Convolution Integral indicates that the process of generation of response in a circuit may be viewed as a process in which the circuit *scans* the input function using its impulse response template as a scanner.

This is the reason why impulse response function is called the *Scanning Function*.

Impulse response of a stable linear time-invariant circuit is a tapering function of time. Therefore, after sufficiently long interval of time after the application of input, we may assume that almost the entire significant content of impulse response function has moved into the input function from left-side of τ-axis. Then, if the input is a step function the output will remain constant with further increase in t since the region of overlap and its area remain unchanging once impulse response has completely *moved into* the input source function. This explains the dc steady-state.

If the input is periodic function, the output also varies periodically with same period once the impulse response has moved into input function completely. This gives rise to periodic steady-state. Sinusoidal steady-state is a special case of periodic steady-state.

Frequency Response Function from Convolution Integral

Let $x(t) = 1\cos\omega t$ be applied to a stable and causal linear time-invariant circuit from infinite past. Then, the response will contain only forced response component. Forced response component is same as steady-state response in the case of a stable circuit . We find the steady-state response by employing convolution integral as below.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} 1\cos\omega\tau \times h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\cos\omega(t-\tau) \ \ [\because x(t)*h(t) = h(t)*x(t)] \\ &= \frac{1}{2}\int_{-\infty}^{\infty} h(\tau)[e^{j\omega(t-\tau)} + e^{-j\omega(t-\tau)}]d\tau \text{ (By Euler's Formula)} \\ &= \frac{e^{j\omega t}}{2}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau + \frac{e^{-j\omega t}}{2}\int_{-\infty}^{\infty} h(\tau)e^{j\omega\tau}d\tau \end{aligned}$$

$$y(t) = \frac{e^{j\omega t}}{2}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau + \frac{e^{-j\omega t}}{2}\left[\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right]^*$$

Let $A\angle\theta = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$; then,

$$y(t) = \frac{A\angle\theta\, e^{j\omega t}}{2} + \frac{A\angle-\theta\, e^{-j\omega t}}{2} = \frac{Ae^{j(\omega t+\theta)}}{2} + \frac{Ae^{-j(\omega t+\theta)}}{2} = A\cos(\omega t+\theta)$$

$\therefore$ Magnitude of frequency response function = magnitude of $\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$

Phase of frequency response function = phase of $\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$

$\therefore$ Frequency Response Function $H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$

We can write $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$ by a simple change of variable.

Sinusoidal steady-state frequency response function $H(j\omega)$ of a linear time-invariant circuit is related to its impulse response h(*t*) by

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

Therefore, we see that, the sinusoidal steady-state frequency response function is completely decided by the impulse response of the circuit. Frequency response function is a disguised version of impulse response function. Convolution integral has assured us that the zero-state response to any arbitrary input can be obtained from impulse response. Then, if impulse response is contained in the frequency response function, it must be possible to determine the zero-state response to arbitrary input from frequency response function too. Or, in other words, if impulse response function is a complete characterization of a linear time-invariant circuit, then, frequency response function must also be an equally complete characterization of the same circuit. The next major part of this text shows that it is.

Example : 12.3-4

The impulse response of a first order circuit is $h(t) = 2e^{-2t}u(t)$. Find its frequency response function.

Solution

The frequency response function

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$

$$= \int_0^{\infty} 2e^{-2t}e^{-j\omega t}\,dt = 2\int_0^{\infty} e^{-(2+j\omega)t}\,dt = \frac{2}{2+j\omega} = \frac{2}{\sqrt{2+\omega^2}}\angle -\tan^{-1}(0.5\omega)$$

Example : 12.3-5

The impulse response of a second order circuit is $h(t) = e^{-2t}\cos 3t\, u(t)$. Find its frequency response function.

Solution

The frequency response function, $H(j\omega)$, is given by

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt = \int_0^{\infty} e^{-2t}\cos 3t\, e^{-j\omega t}\,dt = 0.5\int_0^{\infty} e^{-2t}[e^{j3t}+e^{-j3t}]e^{-j\omega t}\,dt$$

$$= 0.5\int_0^{\infty}[e^{(-2-j(\omega-3))t}+e^{(-2-j(\omega+3))t}]dt = \frac{0.5}{2+j(\omega-3)}+\frac{0.5}{2+j(\omega+3)} = \frac{2+j\omega}{(\omega^2-5)+4j\omega}$$

A Circuit with Multiple Sources – Applying Convolution Integral

The convolution integral gives the response of a circuit to a single input in terms of that input source function and the impulse response function between the point of application of that input and the point at which output is observed. It gives the forced response if input is known to have been applied from $t = -\infty$. And, it gives the zero-state response for a right-sided input. Forced response and zero-state response in a linear time-invariant circuit obey superposition principle for multiple input. Therefore, the response of a linear time-invariant circuit to simultaneous application of two inputs from two different locations in the circuit can be obtained by finding the components of response when each is acting alone and then adding the components. Convolution integrals can be employed to determine the two response components provided we know the response of the circuit to unit impulses applied at the two input locations.

Example : 12.3-6

Find the zero-state response of $v_o(t)$ in the circuit in Fig. 12.3-3 if $v_S(t)$ is a rectangular pulse of 10 V lasting for 1 sec starting from $t = 0$ and $i_S(t)$ is a rectangular pulse of 1.5 A lasting for 1 sec starting from $t = 0$.

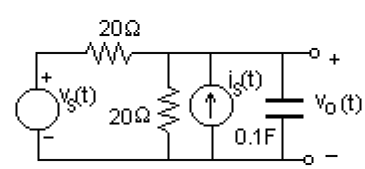

Fig. 12.3-3 Circuit for Example : 12.3-6

Solution

The two circuits for applying superposition principle are shown in Fig. 12.3-4. We need to determine the impulse responses first.

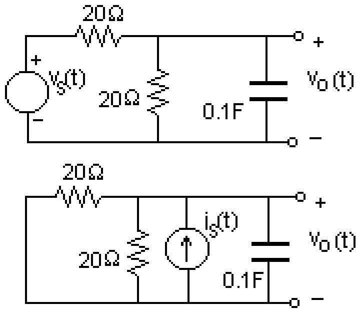

Fig. 12.3-4 Component Circuits for Applying Superposition Principle in Example : 12.3-6

Impulse response in the first circuit – $h_v(t)$

The applied unit impulse voltage gets dropped entirely across the first 20Ω resistor since the capacitor can absorb no part of it. Therefore, $0.05\delta(t)$ current of magnitude 0.05 coul gets dumped into capacitor at $t = 0$. This results in capacitor voltage changing from zero to 0.5 volts at $t = 0$. Therefore, $v_o(0^+) = 0.5$ volts and after that it is a free response of an *RC* circuit with time constant = (20Ω//20Ω)×0.1F = 1 sec. Therefore, $h_v(t) = 0.5\, e^{-t}$ volts.

Impulse response in the second circuit – $h_i(t)$

The applied unit impulse current carrying 1 coul flows entirely through the capacitor, changing its voltage from zero to 10 volts instantaneously at $t = 0$. Therefore, $v_o(0^+) = 10$ volts and after that it is a free response of an *RC* circuit with time constant = (20Ω//20Ω)×0.1F = 1 sec. Therefore, $h_i(t) = 10\, e^{-t}$ volts.

Now the response components can be obtained by convolution integrals.

Response component due to $v_S(t) = v_S(t) * h_v(t)$

$$= \begin{cases} \int_0^t 10\times 0.5e^{-(t-\tau)}d\tau = 5e^{-t}\int_0^t e^{\tau}d\tau = 5e^{-t}[e^t - 1] = 5(1-e^{-t}) \text{ volts for } 0^+ < t < 1 \\ \int_0^1 10\times 0.5e^{-(t-\tau)}d\tau = 5e^{-t}\int_0^1 e^{\tau}d\tau = 5e^{-t}[e^1 - 1] = 8.591e^{-t} \text{ volts for } 1 < t < \infty \end{cases}$$

Response component due to $i_S(t) = i_S(t) * h_v(t)$

$$= \begin{cases} \int_0^t 1.5\times 10e^{-(t-\tau)}d\tau = 15e^{-t}\int_0^t e^{\tau}d\tau = 15e^{-t}[e^t - 1] = 15(1-e^{-t}) \text{ V for } 0^+ < t < 1 \\ \int_0^1 1.5\times 10e^{-(t-\tau)}d\tau = 15e^{-t}\int_0^1 e^{\tau}d\tau = 15e^{-t}[e^1 - 1] = 25.774e^{-t} \text{ V for } 1 < t < \infty \end{cases}$$

The zero-state response when both sources are acting together = sum of the response components.

$$\therefore v_0(t) = \begin{cases} 20(1-e^{-t}) \text{ volts for } 0^+ \le t < 1 \\ 34.365e^{-t} \text{ volts for } 1 < t \end{cases}$$

Zero-Input Response by Convolution Integral

We had observed that convolution integral gives the zero-state response for a right-sided input. That raises the question – can't we bring the zero-input response also under convolution integral?

Zero-input response arises from non-zero initial energy storage in inductors and capacitors. We remember that inductors and capacitors with non-zero initial condition can be replaced by elements with zero initial condition along with suitably connected impulse sources to account for the initial conditions. Thus, we can reduce any circuit problem with non-zero initial energy storage into a zero-state response after replacing the initial voltage across capacitors and initial current through inductors by impulse sources. The resulting circuit will have more than one source. We have seen how response to multiple input can be obtained by convolution integral in the previous sub-section.

However, the zero-state response to initial condition sources does not even require convolution. These are impulse sources. Therefore the convolution is to be done between impulse functions and impulse response functions. The convolution of any function with unit impulse function results in that function itself.

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = \int_{t_-}^{t^+} x(\tau)\delta(t-\tau)d\tau = x(t)$$

Therefore the response to initial condition sources will be scaled impulse response functions relevant to the locations at which the initial condition impulse sources are connected.

Example : 12.3-7

The initial voltage across capacitor in the circuit in Fig. 12.3-3 is –10 volts. Find the total response of the circuit.

Solution

The circuit after replacing –10 V initial condition on capacitor by a impulse current source of magnitude 10 V×0.1 F = 1 coul is shown in circuit (a) of Fig. 12.3-5.

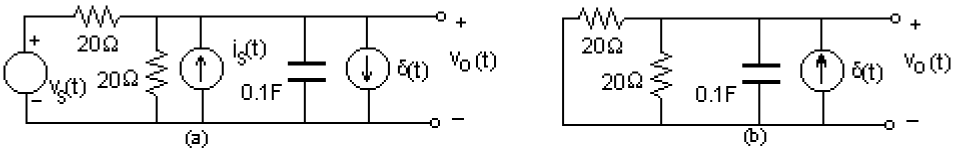

Fig. 12.3-5 (a) Circuit After Replacement of Initial Voltage by Impulse Current Source (b) Circuit for Impulse Response for Current Source Excitation Across Capacitor

This is a three-source problem. We have already solved for zero-state response due to $v_S(t)$ and $i_S(t)$ in Example : 12.3-6. We need to find the impulse response for current source excitation across the capacitor to determine the third component of output. The circuit required for this is shown as (b) in Fig. 12.3-5.

Unit impulse current applied across the capacitor as in (b) of Fig. 12.3-5 flows entirely through the capacitor, changing its voltage from 0 to 1 coul/0.1 F = 10 volts instantaneously at $t = 0$. Thereafter the impulse current source behaves as an open-circuit and the circuit executes its free response. Therefore the required impulse response $h_{ic}(t) = 10\, e^{-t}$ volts.

Now we need to convolve this impulse response with the actual initial condition source function. But the actual initial condition source function is $-\delta(t)$. There is no need to carry out the convolution. The result is known to be $-1 \times h_{ic}(t) = -10e^{-t}$.

The total zero-state response due to the three sources is given by

$$v_0(t) = \begin{cases} 20(1-e^{-t}) - 10e^{-t} = 20 - 30e^{-t} \text{ volts for } 0^+ \le t < 1 \\ 34.365e^{-t} - 10e^{-t} = 24.365e^{-t} \text{ volts for } 1 < t \end{cases}$$

We have accepted the zero-state response for $v_S(t)$ and $i_S(t)$ from Example : 12.3-6 in arriving at this expression.

Since all sources including the initial condition sources are accounted for in this zero-state response, this zero-state response itself is the total response.