Solution

(i) Total current in a parallel combination gets distributed in resistors as per the conductance ratio. The relevant ratio here is 0.1:0.05:0.03:0.02 , *i.e.*, 10:5:3:2. Therefore the currents are 5 A, 2.5 A, 1.5 A and 1 A in 0.1 S, 0.05 S, 0.03 S and 0.02 S resistors respectively.

(ii) The equivalent conductance of a parallel combination is the sum of conductance values of the participating resistors. Hence the equivalent conductance here is 0.2 S. Therefore equivalent resistance is 5 Ω. Therefore the voltage appearing across the current source is 50 V and the power delivered by it is 500 W.

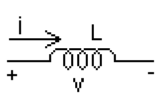
3.2 The Inductor

The physical basis for the two-terminal element called inductor has been dealt with in Chapter 1. We recapitulate briefly.

Moving charges, *i.e.,* current flow, causes a magnetic field to be set up everywhere in space. When the current is time-varying, the magnetic field too will be time-varying. Time-varying magnetic field produces an extra electric force on charges everywhere – extra over whatever other forces that are present. This extra force – induced electric force – is non-conservative and results in an electro-motive force in the circuit. In the case of a closed loop, this electro-motive force is related to the rate of change of magnetic flux by Faraday's law of induction.

This induced electric force will be present everywhere in the circuit when circuit currents are time-varying and, strictly speaking, can not be localised. However, physical devices can be constructed in such a way that the voltage appearing across them when a time-varying current is forced into them is predominantly due to induced electric force alone. If, in addition, the voltage due to induced electric force in this device is much more than such induced voltages elsewhere in the circuit (except in other devices of the same type), then we may localise all the induced voltage in the circuit to voltage drops across the terminals of such devices and assume that there is no induced e.m.f due to time-varying currents in other parts of the circuit. Such a device is an inductor.

Thus, an inductor is a two-terminal physical device intentionally designed to produce a voltage drop that consists of only induced voltage due to time-varying magnetic fields. It is a linear inductor if the magnetic flux linkage in the inductor is proportional to the current producing the flux linkage. The constant of proportionality is the value of inductance of the inductor. We use the same symbol – L – to denote the two-terminal element as well as its inductance, *i.e., L* stands for both the inductor and its inductance value. Its unit will be volt-sec/amp or Weber-Turns/amp, which is given the name 'Henry' and abbreviated as 'H'. The element relation for inductor as per passive sign convention is



$$\psi(t) = L\,i(t) \text{ where } \psi(t) = \text{instantaneous flux linkage in } L$$

$$v(t) = L\frac{di(t)}{dt}\ ;\ i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)\,dt \quad \text{and} \quad \psi(t) = \int_{-\infty}^{t} v(t)\,dt$$

Voltage-current , Flux-linkage-current and Flux linkage-voltage relations for an inductor

We have emphasized the time-varying nature of variables by including (t) in the defining equations. However, we will use the italicized variables without the (t) attached to them also to stand for functions of time. Thus i and $i(t)$ mean the same. We use the second form only when we want to emphasize the dependence on time.

We take up a detailed study of the element relation of an inductor.

The voltage across inductor is proportional to the rate of change of current through it. The current through the inductor is proportional to the area under the voltage waveform, i.e., the volt-sec product (or Wb-T) applied across its terminals from t = -∞ where t = -∞ has to be understood as the moment this inductor came into being.

A restatement of v-i , ψ - v relations for an inductor

These two simple statements have many implications in circuits in which inductor appear. We bring them out in the subsequent sub-sections.

Instantaneous Inductor Current vs. Instantaneous Inductor Voltage

Suppose we know the value of $v(t)$ at some instant $t = t_0$ and let it be v_o. Can we predict the inductor current at that instant ? No, the element relation does not help us there because voltage across inductor depends on rate of change of current and not on current. Therefore, the value of current can be any value at that instant. But the element relation of inductor tells us that, whatever be the value of current at that instant, it must be changing at v_o/L amperes per second rate at $t = t_0$. This implies that, if v_o is positive, the inductor current is on the rise, and, if v_o is negative, it is on the fall. Notice that polarity of v_o reveals the direction of current change and does not reveal anything about the polarity of the current unlike in the case of resistor. The current at $t = t_0$ can be positive or negative quite independent of whether it is increasing or decreasing.

What if v_o is zero? What we can conclude about current in this case will depend on how this zero value was attained by $v(t)$. If $v(t)$ attained this value of zero at $t = t_0$ while it was moving from a negative value to positive value *i.e.*, $v(t)$ was crossing zero in the upward direction, the inductor current will be at a local minimum. If $v(t)$ attained this value of zero at $t = t_0$ while it was moving from a positive value to negative value *i.e.*, $v(t)$ was crossing zero in the downward direction, the inductor current will be at a local maximum. This will become clear if we keep in mind that the derivative of $v(t)$ will decide whether a critical point in $i(t)$ is a local maximum or local minimum. However, if $v(t)$ became zero at $t = t_0$ only because it is identically zero in some interval of time containing this time instant , it will imply that the inductor current is a constant at some value in that interval.

Instantaneous current in an inductor can not be predicted from instantaneous value of voltage across it.

If instantaneous value of voltage is +ve, the inductor current will be increasing at that instant, and, if it is –ve the current will be decreasing at that instant.

When voltage across an inductor crosses zero in the downward direction its current attains a local maximum and when it crosses zero in the upward direction the inductor current attains a local minimum.

Voltage across an inductor carrying a constant current is zero.

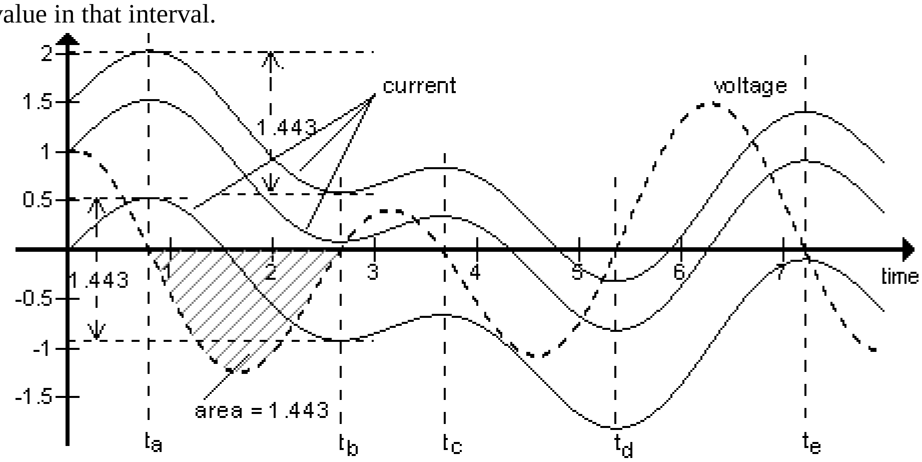


Fig. 3.2-1 Voltage-Current Relation in a 1 H Inductor

Consider the voltage-current relationship for a 1H inductor shown in Fig. 3.2-1. The solid curves represent three possible waveforms of current – all of them will have same first derivative waveform – and the dotted curve shows the applied voltage waveform. The three possible current waveforms are different only by constant values – notice that all three are parallel to each other. Derivative of a constant is zero and hence the voltage appearing across inductor in all the three cases will be represented by the same curve. Also note that at the time instants marked as t_a , t_c and t_e, the $v(t)$ waveform crosses zero in the downward direction, and, $i(t)$ in all the cases attain local maxima at all the three instants. Similarly, at the time instants marked as t_b and t_d, the $v(t)$ waveform crosses zero in the upward direction, and, $i(t)$ in all the three cases reach local minima at those time instants. Further, the polarity of $v(t)$ is negative for all time instants between t_a and t_b, and, we observe that all the three current waveforms decrease in that interval. Similarly, polarity of $v(t)$ is positive for all time instants between t_b and t_c, and, we observe that all the three current waveforms increase in that interval.

Instantaneous current in an inductor can not be predicted from instantaneous value of voltage across it.

If instantaneous value of voltage is +ve, the inductor current will be increasing at that instant, and, if it is –ve the current will be decreasing at that instant.

When voltage across an inductor crosses zero in the downward direction its current attains a local maximum and when it crosses zero in the upward direction the inductor current attains a local minimum.

Voltage across an inductor carrying a constant current is zero.

Change in Inductor Current Function vs. Area under Voltage Function

The relation between current and voltage of an inductor is reproduced below.

$$v(t) = L\frac{di(t)}{dt} \text{ and } i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)\,dt \qquad (3.2\text{-}1)$$

Consider two time instants t_1 and t_2. Applying the integral form of v-i relationship, we get

$$i(t_2) - i(t_1) = \frac{1}{L}\int_{-\infty}^{t_2} v(t)\,dt - \frac{1}{L}\int_{-\infty}^{t_1} v(t)\,dt = \frac{1}{L}\int_{t_1}^{t_2} v(t)\,dt \qquad (3.2\text{-}2)$$

Change in inductor current over a time interval is proportional to area under voltage waveform applied to it during that time interval

Thus the *change* in inductor current is given by *(1/L)* × area under voltage function between the two instants under consideration. This is also expressed as $\Delta i = \frac{\text{volt} - \text{sec}}{L}$ where Δi is the increase in inductor current $i(t)$ over $[t_1, t_2]$ and *volt-sec* is the area under $v(t)$ in the same interval.

Therefore, $i(t_2) = i(t_1) + \Delta i = i(t_1) + \frac{\text{volt} - \text{sec}}{L}$. We can also relate the *volt-sec* product to *change* in flux linkage in the inductor. *In fact, the volt-sec product itself is the change in flux linkage since* $\Delta\psi = L\Delta i$ *= area under voltage function (volt-sec).* Therefore volt-sec and Weber-turns are two units for the same quantity.

Change in flux linkage in an inductor over a time interval is equal to area under voltage waveform applied to it during that time interval

We can calculate only *change* in $i(t)$ given the $v(t)$ unless $v(t)$ is given for $(-\infty, t]$ interval. We can find the absolute instantaneous value of $i(t)$ if we know all the voltage applied to inductor from infinite past to the present instant. However, we need not insist on being given the $v(t)$ from $-\infty$ itself. It is enough that we know the area under $v(t)$ from $-\infty$ to some instant –say $t = t_0$ – and $v(t)$ itself from that instant onwards. This is so because we can split the integral in Eqn. 3.2-2 as shown in Eqn. 3.2-3.

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)\,dt = \underbrace{\frac{1}{L}\int_{-\infty}^{t_0} v(t)\,dt}_{I_{t_0}} + \frac{1}{L}\int_{t_0}^{t} v(t)\,dt \qquad (3.2\text{-}3)$$

The area under voltage waveform applied to an inductor from $t = -\infty$ to $t = t_o$ can be summarized in the form of an initial value for inductor current at $t = t_o$.

Obviously the first term on the right is the inductor current value at t_0. Therefore, we can work out inductor current at an instant if we know its value at some reference instant and the voltage function applied to it from that reference instant onward. This reference instant is usually set as $t = 0$ in analysis of circuits and the value of inductor current at $t = 0$ is termed as initial condition of inductor.

Change in inductor current over $[t_1, t_2]$, Δi = *(Area under inductor voltage over* $[t_1, t_2]$*)/L.*

i(t) at $t = t_2$ *is i(t) at* $t = t_1$ *plus* Δi

$i(t) = I_0 +$ *(Area under inductor voltage over [0, t])/L*

where I_0 *is the current in the inductor at t = 0 and is called initial condition of the inductor .*

Refer to Fig. 3.2-1. The area under $v(t)$ between t_a and t_b is 1.443 volt-sec. The inductance value is 1 H. And, it is shown in the figure that all the three possible $i(t)$ waveforms undergo a change by –1.443 amps in that interval, clearly demonstrating the relation between *change* in inductor current and volt-sec product dumped into the inductor during the relevant time interval. The voltage waveform in the Fig. 3.2-1 is known only for $t \geq 0$. The three current curves shown in that figure represent three possible initial values for the inductor current at $t = 0$. The respective initial current values can be read off the curves at $t = 0$.

Average Applied Voltage for a Given Change in Inductor Current

The amount of current change required in an inductor decides the area-content under voltage waveform to be applied to it to bring about the change.

The time allowed to bring about the change in current decides the average voltage to be applied.

Let us assume that we want to increase the current in an inductor L from I_1 to I_2 ($I_2 > I_1$) in a time interval of Δt. This change may be accomplished by applying any waveform for voltage provided the area under that waveform over Δt is $L(I_2 - I_1)$ volt-sec. This implies that irrespective of the exact waveform of voltage applied its average value over Δt has to be $L(I_2 - I_1)/ \Delta t$ volts.

Now as Δt decreases, *i.e.*, when we try to accomplish the required current change in shorter time interval, the average voltage to be applied increases. Thus, fast current changes in inductor require higher voltage to be applied across it.

Instantaneous Change in Inductor Current

It follows from the last sub-section that the average voltage to be applied to cause a finite amount of change in inductor current increases to infinite value when we try to accomplish the change in current in zero time interval. We can not bring about instantaneous change in inductor current unless we apply or support such an infinite voltage across the inductor.

Let us say we want to change the current in a 0.5 H inductor from 0 to 2 A by applying a rectangular pulse voltage from t = 0. The voltage area content required is 0.5H × 2 A = 1 volt-sec. The height of pulse will depend on the width of pulse. Three cases are shown in Fig. 3.2-2.

When 2.5 V pulse lasting for 0.4 sec is applied, the inductor current ramps up linearly from 0 to 2 A in 0.4 sec with a slope of 5 A/s. When 5 V pulse lasting for 0.2 sec is applied, the inductor current ramps up linearly from 0 to 2 A in 0.2 sec with a slope of 10 A/s. When 10 V pulse lasting for 0.1 sec is applied, the inductor current ramps up linearly from 0 to 2 A in 0.1 sec with a slope of 20 A/s. In all the three cases we have kept the area under the voltage waveform at 1 volt-sec. Now consider further shortening of pulse duration taking it to near-zero width. If we correspondingly increase the pulse height such that the area under the waveform remains at 1 volt-sec always, the change in inductor current will be 2 A always. However, the inductor current waveform will become steeper and steeper till it becomes a vertical waveform as pulse width → 0 and pulse height → ∞. Notice that though pulse height → ∞ as width → 0, its area is constrained to remain 1 volt-sec. Such an idealized waveform with zero width, undefined height and finite area-content of unity is called a *unit impulse function* and denoted by the symbol $\delta(t)$. Its mathematical definition is

An unit impulse waveform can be considered as a limiting case of a rectangular pulse of unit area when its width is sent to infinitesimally small duration

$$\delta(t) = \begin{cases} 0 \ \text{ for } -\infty < t \le 0^- \\ undefined \ at \ t = 0 \\ 0 \ \text{ for } 0^+ \le t < \infty \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

where the time instant $t = 0^-$ is an instant which is arbitrarily close to t =0 but on its left side and time instant $t = 0^+$ is an instant which is arbitrarily close to t =0 but on its right side. Thus the interval $[0^-,0^+]$ is of infinitesimal width ; but 0 comes in the middle of this interval.

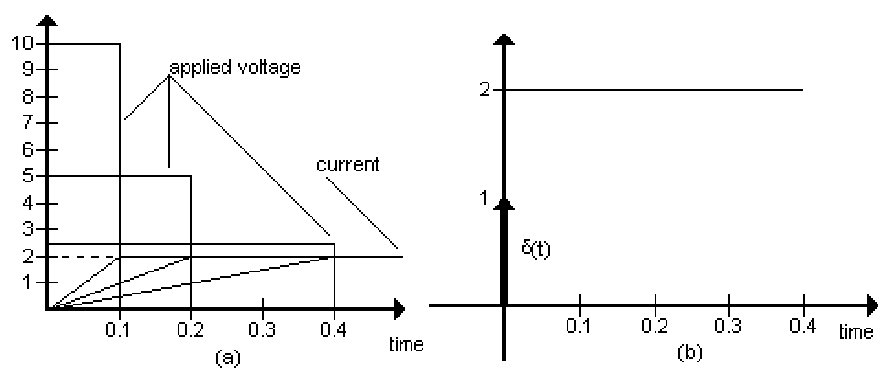


Fig. 3.2-2 Rectangular Pulse Application and Impulse Voltage

The graphical symbol used for $\delta(t)$ is shown in (b) of Fig. 3.2-2. The height of the arrow-terminated vertical line representing $\delta(t)$ *is not* the amplitude of the function (it is *undefined* at that point); rather it indicates the area-content of the waveform. The instantaneous change in inductor current from 0 to 2 A is also shown in the same figure. Now look at the current waveform in the inductor. It is zero till 0^- and 2 A after 0^+ and a discontinuous jump at t =0. This must be 2 times the integral of impulse function (1/L = 2 in this case). Let us verify this.

$$\int_{-\infty}^{t} \delta(t)\, dt = \begin{cases} 0 \text{ for } -\infty < t \leq 0^- \\ \textit{undefined} \text{ at } t = 0 \\ 1 \text{ for } t \geq 0^+ \end{cases} \qquad (3.2\text{-}4)$$

Definition of unit step function and its relation to unit impulse function

This function is defined as a *unit step function* and is denoted by $u(t)$. Thus, when we apply a $\delta(t)$ voltage waveform to an inductor of inductance value L, the current in the inductor jumps up instantaneously *by* 1/L amperes. *Unit impulse voltage source* will dump 1 volt-sec of voltage area content into the inductor instantaneously. Equivalently, *unit impulse voltage source* will dump 1 Weber-turns (Wb-T) of flux linkage into the inductor instantaneously. The result will be a change in its current *by* 1/L amperes.

Strictly speaking, it is the flux linkage in an inductor that can not be changed instantaneously. But in the case of an inductor that is not magnetically coupled to other inductors, this will amount to what we have stated above since flux linkage in such an inductor is proportional to its current. We will modify this statement suitably when we take up the study of coupled circuits later in the book.

Instantaneous change in inductor current

Current in an inductor can not change instantaneously unless an impulse voltage is applied or supported in the circuit.

The current in an inductor *L* changes instantaneously *by* 1/*L* amps when the circuit applies or supports unit impulse voltage across it.

Therefore, if a circuit does not apply or support impulse voltage, the currents in inductors in that circuit will be continuous functions of time.

Inductor with Alternating Voltage Across it

We consider the application of alternating voltage (ac voltage) across an inductor in this sub-section. Alternating voltage is a voltage waveform that alternates between positive and negative voltages *periodically* and has a zero cyclic average. This means that the area under the voltage waveform during positive half-cycle and the area under the voltage waveform during the negative half-cycle are equal. The two half-cycles need not be equal in length. But the net area in a cycle has to be zero. This is equivalent to a *zero dc content* since the dc content of a cyclic waveform is its area-content over a cycle divided by the cycle period. It is possible to express a periodic waveform as a dc term plus a pure alternating term if there is a non-zero dc content in it.

Fig. 3.2-3 shows the results of applying an alternating voltage waveform to an inductor with two values of inductance (1 H and 5 H) considered.

The dotted curve shows the applied voltage and solid curves show the current in the inductor. The integral of applied voltage is also shown in the figure. Both current curves show local maxima and minima at voltage zero-crossing points. The area under one half cycle of voltage is 1 volt-sec and the current in 1 H should change by 1 A over a half cycle and current in 5 H should change by 0.2 A over a half cycle. The Fig. 3.2-3 shows that the current in 1 H inductor varies between 1.4 A and 0.4 A with the initial condition of 0.4 A. The current varies between 0.6 A and 0.4 A in the case of a 5 H inductor with same initial condition.

We need to appreciate three following points in this context.

With a specific area under a half-cycle of voltage waveform, the current in the inductor will change by an amount equal to that area value divided by L. In the next half cycle it will vary by the same amount, but in opposite direction. *Thus, the peak-to-peak value of alternating component of inductor current will be equal to the area of one half-cycle of voltage waveform divided by L.* Therefore, higher the inductance, lower the peak-to-peak ripple current in the inductor. This conclusion is independent of the exact shape of voltage waveform.

If the frequency of voltage waveform is increased without changing its amplitude and waveshape, the half-cycle area decreases due to reduction in half cycle duration. Then, the peak-to-peak ripple current will also decrease. *Therefore, higher the frequency of alternating voltage applied to an inductor, lower the peak-to-peak amplitude of the alternating component of inductor current.*

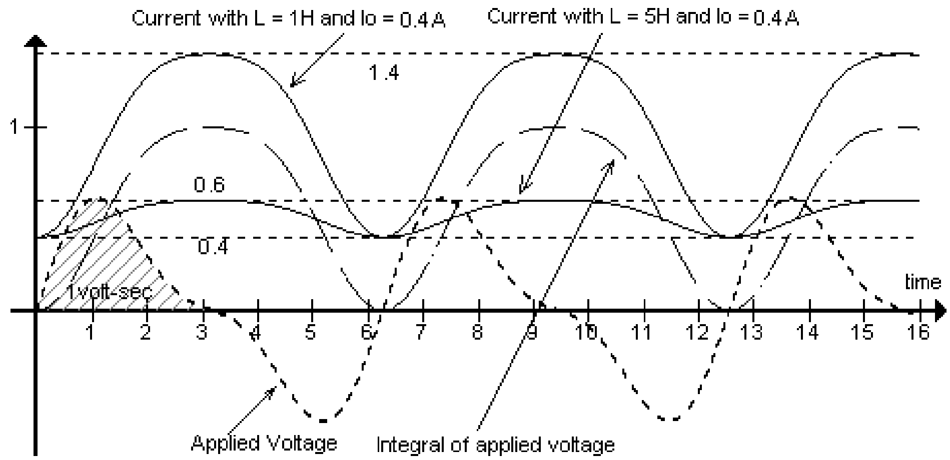


Fig. 3.2-3 Alternating Voltage Application to an Inductor

> There can be a dc current through an inductor even when the applied voltage waveform is a pure alternating one. The amount of dc content depends upon the initial condition of the inductor and the instant at which the voltage waveform is switched on to the inductor.

The dc content in inductor current is decided by two factors – the initial condition and the instant of application of the alternating voltage. Examine the integral of voltage waveform in Fig. 3.2-3. The voltage waveform was applied to the inductor at its zero-crossing. Therefore, its integral goes to a maximum value of 1 volt-sec in the first half-cycle and then returns to zero at the end of second half-cycle. It does not go negative. This area waveform divided by L will give us the current in the inductor with zero initial condition. Notice that that current will have a dc content since the voltage area waveform has a dc content. *Thus, the net dc content in the inductor current will be its initial condition value plus cyclic average of voltage area waveform divided by L.* Notice that the second contribution to dc content in the inductor current will depend on at which point in the voltage waveform we start applying it to the inductor. There will exist one particular waveform position in any periodic voltage such that the dc contribution to inductor current will be zero if switching is done at that position.

When the applied voltage across an inductor is a periodic alternating waveform, the current in the inductor will contain an alternating component with the same period. The peak-to-peak amplitude of this alternating component will be directly proportional to half-cycle area of voltage waveform and inversely proportional to inductance value. It decreases with increase in frequency of the voltage.

Therefore a large valued inductor in a circuit can absorb alternating voltages in the circuit without contributing significant amount of alternating currents to the circuit.

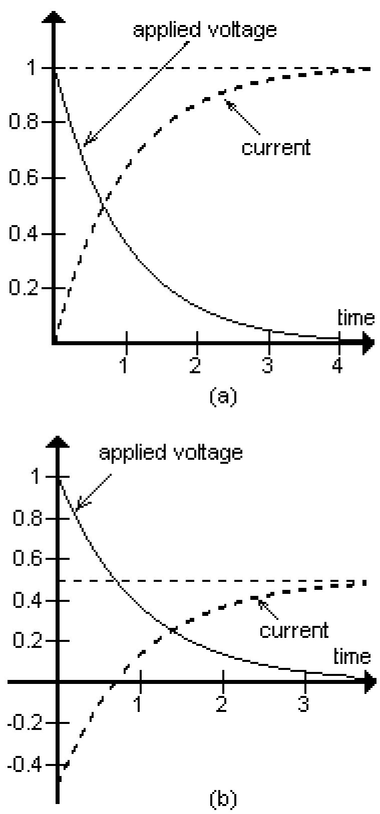


Fig. 3.2-4 Inductor with Exponential Voltage Applied to It

Inductor with Exponential and Sinusoidal Voltage Input

Consider an inductor of 1 H with a voltage $v_S(t) = e^{-t}$ volts switched on to it from $t = 0$ onwards. We do not know what voltage was applied to inductor till t=0. But the net effect of all that voltage which may have been applied to the inductor is condensed in the initial condition available to us. We assume an initial current of zero in (a) of Fig. 3.2-4 and initial current of –0.5A in (b) of the same figure. Straightforward integration gives us $i(t) = I_0 + (1\text{-}e^{-t})$ as the current in 1H inductor. The applied voltage and current for the two initial condition values are shown in Fig. 3.2-4.

The inductor preserves the waveshape of the input except for a dc offset, *i.e.*, the current in the inductor is an exponential of same index as that of the applied voltage. This is due to the fact that exponential function does not change its shape on differentiation and integration. A sinusoidal function too has that property. Therefore, we expect the inductor current to be a sinusoid of same frequency as that of input when a sinusoidal voltage is applied to it. There may be dc offset in the current in Fig. 3.2-5 shows the current in a 1H inductor with zero initial current when (a) sin t is applied (b) sin $(t + \pi/4)$ is applied and (c) sin $(t + \pi/2)$ is applied.

The alternating component of current indeed preserves the waveshape of input. The dc offset is entirely due to the input since initial current was stated to be zero. Notice that when a sinusoidal voltage is switched on at its zero-crossing the resulting current is unipolar and reaches a peak which is twice that of its ac component amplitude, *i.e.,* it has a dc content equal to ac amplitude. This is of course clearly seen by simple integration of $v_S(t)$ as below.

$$\begin{aligned} i(t) &= I_0 + \frac{1}{L}\int_0^t \sin(\omega t+\theta)\,dt \\ &= I_0 + \frac{1}{\omega L}\Big[-\cos(\omega t+\theta)\Big]_0^t \\ &= I_0 + \frac{1}{\omega L}[\cos\theta - \cos(\omega t+\theta)] \end{aligned}$$

$I_0 = 0$ A , $\omega = 1$ rad/sec and $L = 1$ H for the waveforms shown Fig. 3.2-5. Both the waveforms and the above equation makes it clear that dc offset in the inductor current will be zero (if initial condition is zero) if the sinusoidal voltage is switched on at its peak.

Sinusoidal voltage is a special case of a general periodic alternating waveform. We expect the current peak-to-peak amplitude to go down with the frequency. The above equation shows that it does so. Moreover, when voltage across the inductor is a sine wave, its current waveform will be an inverted cosine wave. Corresponding positions in an inverted cosine wave will take place after $T/4$ seconds with respect to the sine wave. For example, the positive peak of current comes after $T/4$ sec in time axis or $\pi/2$ rad in ωt axis the positive peak of voltage appears.

Inductor preserves the waveshape for exponential and sinusoidal inputs.

The amplitude of current sinusoid in an inductor is inversely proportional to the product of frequency of applied voltage sinusoid and inductance value.

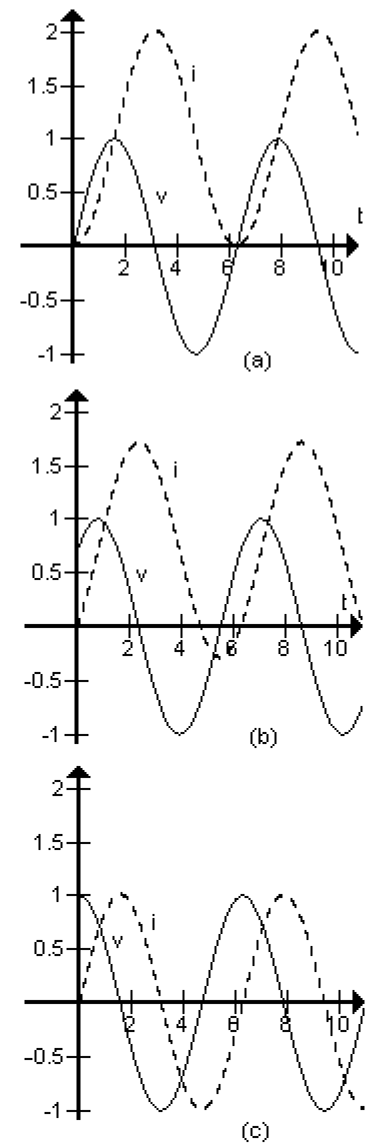


Fig. 3.2-5 Inductor with Sinusoidal Voltage Applied to It

Linearity of Inductor

The flux linkage in an inductor and the current through it are related by a simple proportionality relationship. Hence ψ versus i curve for a linear inductor is a straight line passing through origin. In that sense inductor is linear.

However, that is not what we mean when we say an electrical element is a linear element. *We call an electrical element linear when its voltage-current relationship satisfies two principles – the principle of homogeneity and the principle of additivity.* Principle of homogeneity requires that when input is scaled by a real constant the output also must get scaled by the same constant.

We will treat $i(t)$ as input and $v(t)$ as output first. This implies that we are applying a current source across inductor and observing the voltage appearing across the combination as output. The governing equation then is $v(t) = L\ di(t)/dt$. Obviously, when $i(t)$ is multiplied (scaled) by a real constant α , $v(t)$ also gets scaled by same number. Hence principle of homogeneity is satisfied.

Principle of additivity requires that when two inputs are applied simultaneously, the output observed is the sum of individual outputs observed when these inputs are applied individually. Let us say the voltage across inductor is $v_1(t)$ when a current source of $i_1(t)$ is applied to it and voltage across inductor is $v_2(t)$ when a current source of $i_2(t)$ is applied to it. Then the voltage will be $v_1(t) + v_2(t)$ when $i_1(t) + i_2(t)$ is applied if principle of additivity is satisfied. Obviously this too is true in the present case.

Therefore the relation $v(t) = L\dfrac{di(t)}{dt}$ satisfies both principles.

Now we consider voltage as input and current as output. This implies that we are applying a voltage source across the inductor and observing its current as output. The governing relationship in this case is

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)\ dt$$

It may easily be verified that this relationship satisfies both the requirements. However, there is a caveat here. *We do not know v(t) for t* $<$ 0. Therefore we write this relationship as

$$i(t)=\frac{1}{L}\int_{-\infty}^{t}v(t)\ dt\ =\ \frac{1}{L}\int_{-\infty}^{0}v(t)\ dt\ +\frac{1}{L}\int_{0}^{t}v(t)\ dt\ =I_0\ +\frac{1}{L}\int_{0}^{t}v(t)\ dt$$

thereby absorbing all of $v(t)$ for $t < 0$ into a single number I_0 .

Now if the portion of $v(t)$ that we apply *i.e.*, for $t \geq 0^+$ is multiplied by a real constant α, the inductor current will be

$$i(t) = I_0\ +\frac{\alpha}{L}\int_{0}^{t}v(t)\,dt$$

and that is not α times the earlier current. Hence principle of homogeneity is not satisfied by the total current unless the initial condition is zero. Therefore, we will have to make a qualified statement that the principle of homogeneity is satisfied by the component of current contributed by the applied voltage function. Similarly,

$i(t)$ when $v_1(t)$ is applied $i_1(t) = I_0\ +\frac{1}{L}\int_{0}^{t}v_1(t)\,dt$

$i(t)$ when $v_2(t)$ is applied $i_2(t) = I_0\ +\frac{1}{L}\int_{0}^{t}v_2(t)\,dt$

$i(t)$ when $v_1(t)+v_2(t)$ is applied $i_{12}(t) = I_0\ +\frac{1}{L}\int_{0}^{t}\left(v_1(t)+v_2(t)\ \right)dt$

$$= I_0\ +\frac{1}{L}\int_{0}^{t}v_1(t)\,dt+\frac{1}{L}\int_{0}^{t}v_2(t)\,dt\ \neq i_1(t)+i_2(t)$$

Hence the principle of additivity also is not satisfied by the total current unless the initial condition is zero. Therefore, we will have to make a qualified statement that principle of additivity is satisfied by the component of current contributed by applied voltage function.

These two principles put together is called superposition principle. *A linear element is one that satisfies superposition principle.* Inductor is a linear element if it is understood that the superposition principle has to be applied to the current component which is produced by the applied voltage from t = 0 onwards. The initial current has to be excluded from the purview of superposition principle.

An inductor with zero initial current is a linear electrical element. An inductor with non-zero initial current is a linear element as far as the current component caused by applied voltage is concerned.

Energy Storage in an Inductor

An arbitrary $v(t)$ is applied across an inductor from t = 0 causing a current $i(t)$ through it. The source delivers power and energy to the inductor in this process. We will derive an expression for energy delivered to the inductor as a function of time, *i.e.*, $E_L(t)$ and show that this energy is not dissipated by the inductor but stored by it.

$$E_L(t) - E_L(0) = \int_{0}^{t}v(t)i(t)\,dt = L\int_{0}^{t}i(t)\frac{di(t)}{dt}\,dt = L\int_{0}^{t}i(t)\,di(t) = \frac{L}{2}\int_{0}^{t}d\left[i(t)\right]^2 = \frac{\left[i(t)\right]^2-\left[i(0)\right]^2}{2L}$$

$$\therefore E_L(t) = \frac{1}{2}L\left[i(t)\right]^2$$

Therefore the change in energy delivered to an inductor over a time interval is the difference between the quantity $(1/2)Li^2$ *evaluated at the end of the interval and at the beginning of the interval. The energy delivered till t is given by* $(1/2)Li^2$ *where i is the value of current at t.*

Assume that we applied some $v(t)$ to L from t =0 to t_0 and that initial condition of inductor was 0 amp. At t_0 the current is I. Energy delivered to the inductor up to t_0 is $LI^2/2$ and all of it was delivered by the source connected since inductor had zero energy initially. At t_0 we remove the voltage source and short the inductor. That makes the voltage across inductor zero and therefore its current will continue at I. The power into or out of the inductor is zero since voltage across it is zero. Therefore it will neither deliver nor take energy as long as it is kept shorted.

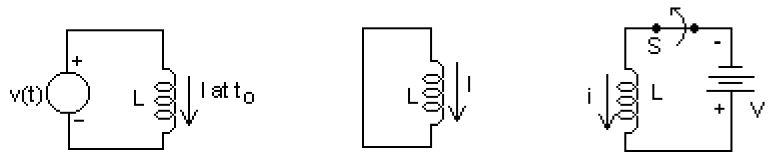


Fig. 3.2-6 On Energy Storage in an Inductor

After some time we remove the short-circuit and connect the inductor to a dc voltage source of value V as shown in third circuit in Fig. 3.2-6. Notice that the polarity of applied voltage V is such that the current in the inductor goes down linearly with a slope of V/L amps/sec starting from I. It will take LI/V seconds for the current to reach zero. The switch S is a zero-current sensing switch and goes open at the instant the circuit current touches zero thereby isolating the inductor from any further change in current. Let us calculate the energy delivered to the dc source in this process.

The total energy delivered to an inductor carrying a current I is $(1/2)LI^2$ Joules. That this energy is stored in the magnetic field of the inductor is established here

$$i(t) = I - \frac{V}{L}t \text{ ; } t \text{ measured from the instant at which V was connected to L}$$

$$\text{Energy delivered to DC Source} = \int_0^{LI/V} Vi(t)dt = \int_0^{LI/V} \left(VI - \frac{V^2 t}{L}\right)dt = \frac{1}{2}LI^2$$

Hence, we see that the energy *delivered* by inductor *to* the dc source in this circuit is exactly equal to the energy *delivered by* the voltage source $v(t)$ *to* the inductor in the first circuit. Where was this energy when the inductor was in the second circuit? Therefore,

The total energy delivered to an inductor carrying a current I is (1/2)LI^2 Joules and this energy is stored in its magnetic field. The inductor will be able to deliver this stored energy back to other elements in the circuit if called upon to do so.

Example : 3.2-1

The current in an inductor of 2H is shown in Fig. 3.2-7. $v(t)$ does not contain any impulse. (i) What is the initial condition for inductor? (ii) Obtain $v(t)$ for 0 to 9 s and plot it. (iii) Obtain the function $p(t)$ – the power delivered to the inductor – and plot it. (iv) Obtain the function $E(t)$ – the energy stored in the inductor – and plot it. Identify the time intervals during which the voltage source charges the inductor and discharges the inductor.

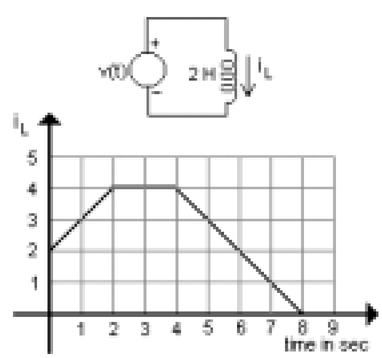


Fig. 3.2-7 Circuit and Waveform for Example : 3.2-1

Solution

Only an impulse voltage can change the inductor current over $[0^-,0^+]$. The current at $t = 0^+$ is read from the given i_L waveform as 2A. ∴Initial current in 2H inductor at $t = 0^-$ = 2A

We use $v(t) = L(di/dt)$ to work out the $v(t)$ waveform. $i(t)$ is a piecewise linear function. The slope of current is 1A/s in the (0,2) interval, 0 A/s in (2,4) interval and –1A/s in the (4,8) interval. Note that all intervals are open intervals. This is so because the current waveform is not differentiable at the end points of the intervals and hence the endpoints will have to be excluded from the domain of the derivative function. We expect to observe jump discontinuities at those points in the $v(t)$ function. Using the slope values we can plot the function $v(t)$ as in Fig. 3.2-8 .

The $v(t)$ waveform is discontinuous pulse waveform as expected. Notice that inductor accepts a *discontinuous voltage input* and generates a *continuous current* in response. This illustrates the ability of the inductor to keep a circuit current smooth.

The power and energy waveforms are calculated by

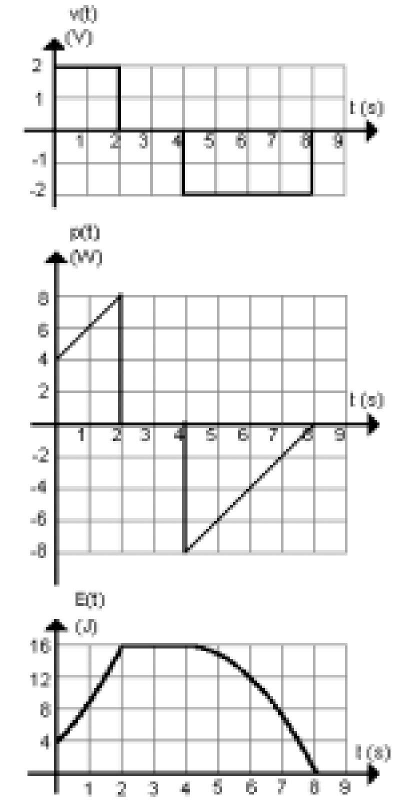


Fig. 3.2-8 Voltage, Power and Energy Waveforms in 2H Inductor in Example : 3.2-1

$$p(t) = v(t) i_L(t)$$

$$E(t) = E(0^+) + \int_{0^+}^{t} p(t)\,dt$$

0^+, 0^- and 0 are to be treated differently if there is an impulse voltage at $t = 0$. Generally speaking, the right instant to use in the energy equation is 0^+. If there is an impulse present at $t = 0$ it has to be accounted by suitably modifying the initial condition. We should avoid trying to integrate the product of impulse and a step discontinuity! That is why we use 0^+ as the lower limit of integration in the energy function. Impulse, if present, will result in a sudden change in initial condition over $[0^-, 0^+]$. We calculate the new initial condition at $t = 0^+$ and then evaluate the initial energy $E(0)$ as $0.5Li(0^+)^2$.

The power waveform in Fig. 3.2-8 shows that the source delivers power to the inductor during (0,2s) interval and source accepts power from inductor during (4s, 8s) interval. Inductor can not dissipate energy. It can only store it. Therefore, when source delivers power to it, the energy storage in the inductor increases – this is what we call *charging* an inductor. The opposite process in which the inductor delivers energy to some other element thereby reducing its stored energy level is called *discharging* an inductor. Thus, in this example, the voltage source charges the inductor during (0,2s) interval and discharges the inductor during (4,8) interval.

The inductor had 4 Joules of energy to begin with. The source delivered 12 Joules of energy to it over (0,2s) interval, raising its energy storage to 16 Joules. The inductor gave all of 16 Joules to the source over (4s, 8s) interval and settled down at zero energy level. Therefore the source received a net energy of 4 Joules from the inductor.

Example : 3.2-2

The initial current in the 0.5 H inductor in Fig. 3.2-9 was 1A at $t = 0^-$. Find the applied voltage $v(t)$ for 0 to 9sec if the i_L waveform is as shown in Fig. 3.2-9.

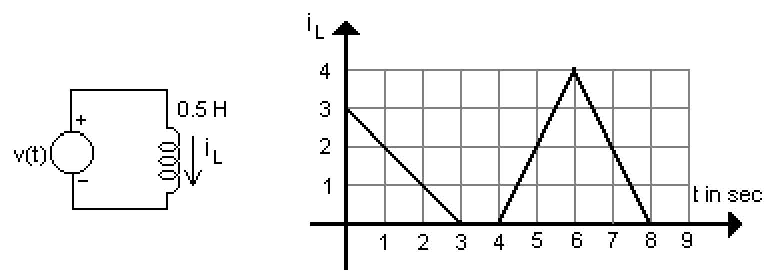


Fig. 3.2-9 Circuit and Waveform for Example : 3.2-2

Solution

The value of current at $t = 0^+$ is different from the given initial condition of 1 A at $t = 0^-$. The current at $t = 0^+$ is seen to be 3 A from the given data. Hence the flux linkage of inductor seems to change instantaneously from 0.5×1 Wb-T to 0.5×3 Wb-T *i.e.,* a change by +1 wb-T. This can happen only if a voltage area-content of 1 volt-sec gets dumped into the inductor at $t = 0$ instantaneously. Only a voltage impulse function of magnitude equal to 1 volt-sec can do this. Remember that *magnitude* of *impulse* is its area-content. Therefore $v(t)$ should contain $\delta(t)$.

The current starts decreasing after $t = 0^+$ at the rate of –1 A/s and continues to fall at that rate in the interval (0,3s) till it reaches zero value.

After 3 sec it remains quiescent at 0 A. After that it rises with a rate of 2 A/s in (4,6) interval and then falls at a rate of –2 A/s in (6,8) interval. Multiplying the various values of slope of current with inductor value, we get the voltage waveform as in Fig. 3.2-10. Notice the impulse function at $t = 0$.

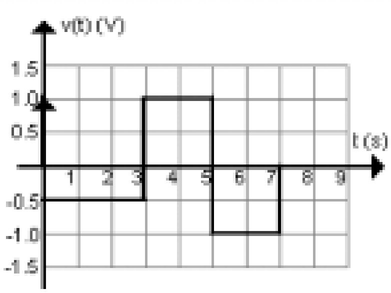


Fig. 3.2-10 Voltage Waveform Across 0.5 H Inductor in Example : 3.2-2

Example : 3.2-3

The current through a 1H inductor is shown in Fig. 3.2-11 for 0 to 18 sec interval. The applied voltage is known to be impulse-free. (i) What is the initial current in the inductor at $t = 0^-$? (ii) Obtain and plot the applied voltage $v(t)$ and the power delivered to the inductor, $p(t)$? (iii) What is the net energy delivered by the voltage source to the inductor?

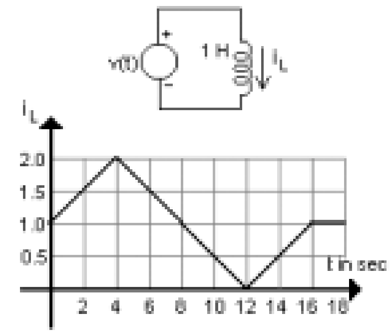


Fig. 3.2-11 Circuit and Waveform for Example : 3.2-3

Solution

The initial current at $t = 0^-$ is 1 A since only an impulse voltage can change the initial current instantaneously.

The value of *di/dt* is 0.25 A/s in (0,4s) interval, -0.25 A/s in (4s,12s) interval, 0.25 A/s in (12s,16s) interval and 0 A/s in (16s,18s) interval. Therefore $v(t)$ is 0.25 V in (0,4s) interval, -0.25 V in (4s,12s) interval, 0.25 V in (12s,16s) interval and 0 Vin (16s,18s) interval. The $v(t)$ waveform is shown in Fig. 3.2-12 .

The power waveform is obtained by taking $v(t)\ i_L(t)$ product and plotting it. This is also shown in Fig. 3.2-12.

The inductor started with 1 A at $t = 0^+$ and ended with 1 A at 16 sec. Therefore the net change in stored energy of the inductor is zero. Hence the net energy delivered by the voltage source to inductor must also be zero.

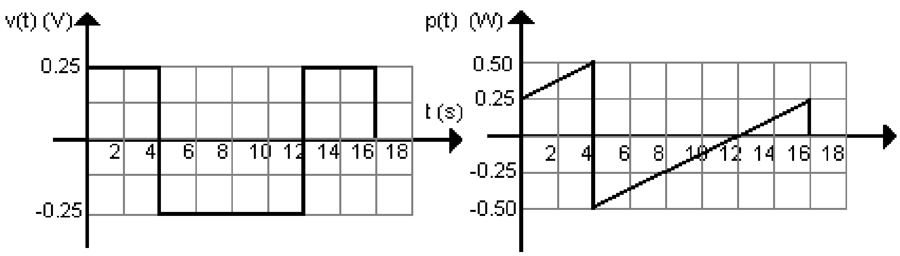


Fig. 3.2-12 Voltage and Power waveforms for Example : 3.2-3

Example : 3.2-4

The voltage waveform applied to an inductor of 2 H is shown in Fig. 3.2-13. (i) Find the magnitude of current in the inductor at t = 9 s if the initial current is zero. (ii) What must have been the initial current if the current at 9 sec is found to be 0 amps? (iii) Plot current in the inductor in the first case.

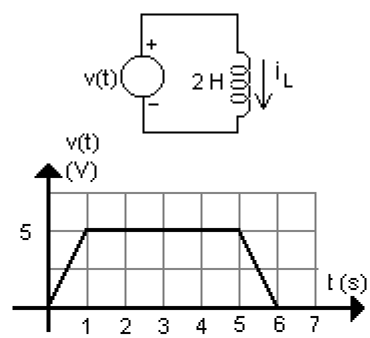


Fig. 3.2-13 Circuit and Waveform for Example : 3.2-4

Solution

(i) Inductor current at any instant is given by its initial current plus the volt-sec product divided by inductance value where volt-sec is the area under the voltage curve from initial instant to the instant at which the current is calculated. The initial instant is 0 A here. Final instant is 9 sec. The volt-sec available in this range is 25 volt-sec. Inductance value is 2 H. Initial current is 0. Therefore inductor current at $t = 9$ sec is 12.5 A. In fact it is 12.5 A from t=6 sec onwards.

(ii) 25 Wb-T of additional flux linkage will change the current of a 2 H inductor *by* 12.5A. Hence if current at $t = 9$sec is observed to be 0 A, then the initial current must have been –12.5 A.

(iii) The equations for $v(t)$ in various time intervals are shown below.

$$v(t) = \begin{cases} 5t \text{ for } 0 \le t < 1 \\ 5 \text{ for } 1 \le t < 5 \\ 5 - 5(t-5) \text{ for } 5 \le t < 6 \\ 0 \text{ for } 6 \le t \le 9 \end{cases}$$

The inductor current can be found by integrating the $v(t)$ expression and dividing the integral by L.

$$i(t)=\begin{cases}0+1.25t^2 \text{ for } 0\le t<1\\1.25+2.5(t-1) \text{ for } 1\le t<5\\11.25+2.5(t-5)-1.25(t-5)^2 \text{ for } 5\le t<6\\12.5 \text{ for } 6\le t\le 9\end{cases}$$

This is plotted in Fig. 3.2-14.

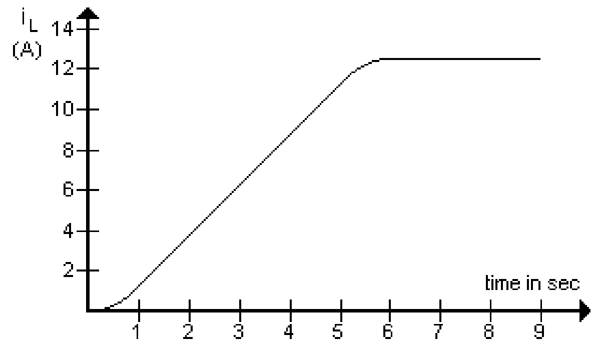


Fig. 3.2-14 Plot of Current in 2H Inductor in Example : 3.2-4

Example : 3.2-5

The pulse voltage waveform shown in Fig. 3.2-15 is applied to an inductor of 0.5 H with initial current of 1 A. (i) Find the inductor current and energy stored in the inductor at (a) t = 1 s (b) t = 2 s and (c) t=3 s. (ii) What is the net energy delivered to inductance in the time interval (a) 0 to 1 sec (b) 1sec to 2 sec and (c) 0 to 2 sec? (iii) What is the maximum value of inductor current and when does it occur?

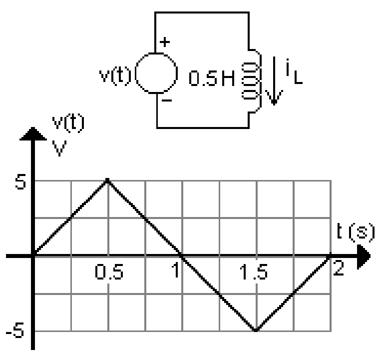


Fig. 3.2-15 Circuit and Waveform for Example : 3.2-5

Solution

(i) The area under voltage waveform from 0 sec to 1 sec is 2.5 volt-sec. The value of inductance is 0.5 H. Therefore the *change* in inductor current over this 1 sec interval is 2.5/0.5 = 5 A. Initial current was 1 A. There is no impulse voltage present at t = 0. Therefore there is no instantaneous change in current at t = 0. Hence the value of current at t = 1 s is 1+5 = 6 A.

The area under voltage waveform from 0 s to 2 s is zero. Therefore the net *change* in inductor current over this interval is zero. Hence current at t = 2 s is 1+0 = 1 A.

Only zero voltage is applied after 2 sec. Hence the area under voltage waveform from 0 s to 3 s is again zero and hence current at t = 3s is 1 A. In fact, current is at 1 A from 2 s onwards.

Energy storage in an inductor is given by $0.5Li^2$ Joules. Hence energy storage in 0.5 H inductor at t = 1 s is 9 J , at t = 2 s is 0.25 J and at t = 3 s is 0.25 J.

(ii) Net energy delivered to inductor in the first 1 second = Energy storage in the inductor at t = 1 s minus energy storage in it at t = 0 s = 9 – 0.25 = 8.75 J

Net energy delivered to inductor in the interval between 1 s and 2 s = Energy storage in the inductor at t = 2 s minus energy storage in it at t = 1 s = 0.25 – 9 = -8.75 J

Net energy delivered to inductor in the interval between 0 s and 2 s = Energy storage in the inductor at t = 2 s minus energy storage in it at t = 0 s = 0.25 – 0.25 = 0 J

(iii) The area under voltage waveform keeps increasing in the interval [0,1] and hence inductor current keeps increasing in this interval. The area under voltage waveform starts decreasing after 1 s since voltage becomes negative from that point. Hence t = 1 s is a time point at which the inductor current ceases to increase and starts to decrease. Hence it must reach a local maximum there. It is a global maximum as well since the voltage does not become positive again. Value of this maximum current is 6 A. It occurs at t = 1 s.

Example : 3.2-6

The initial current in the inductor in the circuit in Fig. 3.2-16 is 0.2 A in the direction shown. What should be V if the current in the inductor is to become 0 amp at t= 5 ms?

Solution

The area under voltage waveform from 0 s to 2 ms is 0.02 volt-sec. The value of inductance is 0.2 H. Hence the *change* in inductor current over the first 2 ms will be 0.02/0.2 = 0.1 A. Therefore the current in 0.2 H inductor at t = 2 ms will be 0.2 + 0.1 = 0.3 A.

-V volts is applied for 1 ms and 0 volts is applied thereafter. Therefore, volt-sec added to the inductor will be –0.001V for all $t \geq 3$ ms. We want the current to become zero at 5 ms. This is possible only if it already becomes zero at 3 ms due to the –0.001V volt-sec dumped into the inductor during 2 ms to 3 ms interval.

The required flux linkage *change* is - 0.2 H × 0.3 A = - 0.06 Wb-T. *Change* in flux linkage in an inductor is equal to the volt-sec dumped into it. Therefore V has to be 0.06/0.001 = 60 volts.

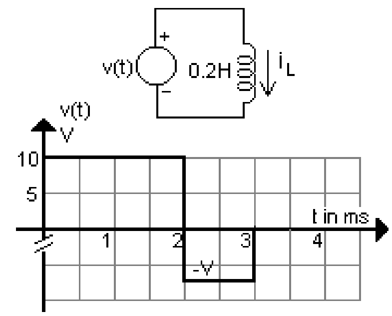


Fig. 3.2-16 Circuit and Waveform for Example : 3.2-6

Example : 3.2-7

An arbitrary time-varying voltage waveform is applied across 1.5 H with zero initial conditions as in Fig. 3.2-17. The current through the inductance is found to be 7.5 A at 7 s. (i) What must be the dc voltage that should be used to replace this source such that the current through the inductor will be the same at 7 s? (ii) If this replacement is carried out will the current in the inductor be same in the two cases at t = 9 s?

Solution

(i) The *change* in inductor flux linkage over any time interval is equal to the area under voltage waveform during that interval. The *change* in flux linkage over 7 s interval is 1.5 H × (7.5 – 0) = 11.25 Wb-T. Therefore the volt-sec during that period must be 11.25 volt-sec. If a constant voltage is to provide this much volt-sec in 7 s its value must be 11.25/7 = 1.607 V.

(ii) If this replacement is carried out, the inductor current will be same in the two cases at 7 s. That is all we can assert from the data provided. Since the $v(t)$ waveform is unknown we can not expect its area at any particular time instant to be equal to the area under a constant function. There is no reason why it can not be so too. In short, in the absence any additional data on $v(t)$ we can not make any predictions about equality of currents at 9 sec in the two cases.

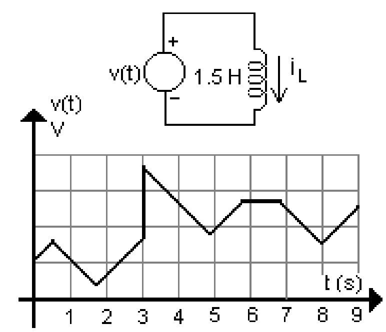


Fig. 3.2-17 Circuit and Waveform for Example : 3.2-7

Example : 3.2-8

A periodic voltage waveform is applied across an inductor of value 0.1 H from t = 0 s as in Fig. 3.2-18 . The current in the inductor is found to vary periodically between 1 A and 5 A. (i) What is the full-cycle average value of the applied voltage waveform? (ii) What is the half-cycle average of the applied voltage waveform? (iii) What was the initial current in the inductor? (iv) Find V_p.

Solution

(i) The current is stated to be periodic. Therefore it must either be a pure alternating waveform or such an alternating waveform plus a dc offset. Differentiation of a pure alternating waveform gives another pure alternating waveform. Differentiation of a dc term can give only zero. Hence the derivative of inductor current will not contain dc term. Derivative of current multiplied by inductance value is the voltage across the inductor. Therefore voltage across the inductor will not have a dc value. But the dc content in a periodic waveform is nothing but its average over a cycle period. Therefore this voltage waveform has a full-cycle average of 0 volts.

(ii) Half-cycle average of alternating voltage = Half-cycle area / half the time period. Half-cycle area of the alternating voltage is given by *change* in flux linkage of inductor between the maximum and minimum current values. It is (5-1) A × 0.1 H = 0.4

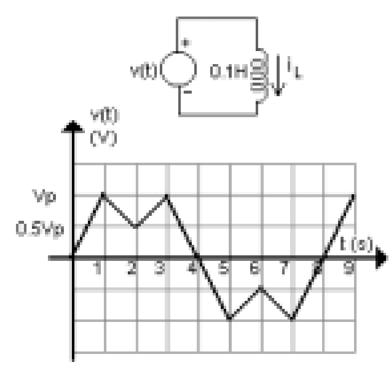


Fig. 3.2-18 Circuit and Waveform for Example : 3.2-8

Wb-T in this case. Therefore the half-cycle area of voltage waveform is 0.4 volt-sec and its half-cycle average value is 0.4 volt-sec /4 sec = 0.1 volts.

(iii) The inductor current is periodic between 1 A and 5 A. There is no impulse content in the applied voltage. Hence its initial current must have been 1 A.

(iv) The half-cycle area in terms of V_p is = $(0.5\ V_p + 0.5\ V_p + 0.25\ V_p)\times 2 = 2.5\ V_p$ volt-sec. This must be equal to 0.4 volt-sec.

$\therefore\ V_p = 0.4/2.5 = 0.16$ volt.

3.3 Series Connection of Inductors

A single equivalent inductor can replace many inductors connected in series for specific analysis purposes. We look into series equivalent and constraints on it in this section.

Series Connection of Inductors with Same Initial Current

We consider series connection of n inductors *which have no mutual magnetic coupling* among them. Let the inductance values be L_1, L_2 ..... L_n. We assume that they have the same initial current in the same direction at $t = 0^-$. Let the applied voltage be $v(t)$ and the current in the series combination be $i(t)$. See Fig. 3.3-1.

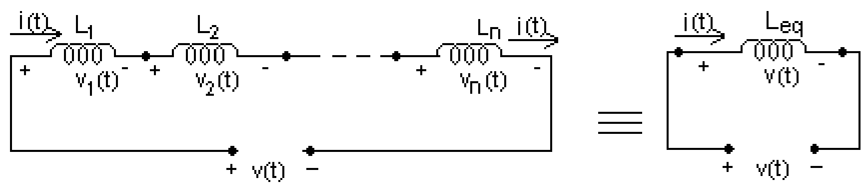


Fig. 3.3-1 Series Connection of 'n' Inductors

Applying KVL along with the element equation of inductor, we get

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) + \cdots\cdots + v_n(t) \\ &= L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + \cdots\cdots + L_n \frac{di(t)}{dt} \\ &= (L_1 + L_2 + \cdots\cdots + L_n)\frac{di(t)}{dt} \\ &= L_{eq}\frac{di(t)}{dt} \text{ where } L_{eq} = L_1 + L_2 + \cdots\cdots + L_n \end{aligned}$$

Series Connection of Inductors

Series connection of many inductors without mutual coupling and with same initial currents can be replaced by a single inductor of inductance value equal to sum of inductance values of all the inductors in series.

Total voltage in a series combination divides in proportion to inductance value across various inductors.

Total flux linkage developed in a series combination is distributed in proportion to inductance value in various inductors.

Thus, a series connection of n inductors may be replaced by an equivalent inductor with an inductance value equal to sum of the inductance values of n inductors as far as the v-i relationship is concerned. The total applied voltage across the combination is shared by the various inductors in direct proportion to inductance value. *i.e.*,

$$\begin{aligned} & v(t) = v_1(t) + v_2(t) + \cdots\cdots + v_n(t) \\ & v_1(t) : v_2(t) : \cdots\cdots : v_n(t) = L_1 : L_2 : \cdots\cdots L_n \\ & v_j(t) = \frac{L_j}{L_{eq}} v(t) \ \text{ for } \ j = 1 \cdots n \end{aligned}$$

It can be seen that sum of flux linkages in the individual inductors is same as the flux linkage of the equivalent inductor. Thus flux linkage is shared in proportion to the inductance value. Similarly the total energy stored in all inductors put together is the same as the energy storage calculated using equivalent inductance value.

Thus series equivalent of n inductors is 'equivalent' with respect to v-i relation, flux linkage and stored energy. But keep in mind that the participating inductors should not have mutual magnetic coupling. And that they must have the same initial current.

Series Connection with Unequal Initial Currents

What happens if they have different initial currents? We take up a simple situation of two inductors –L_1 and L_2 – in series with initial currents of I_1 and I_2 respectively. They have been connected in series at $t = 0$ and the terminals of the series