

4.5 Nodal Analysis of Circuits Containing Dependent Current Sources

Dependent current sources do not pose any problem for nodal analysis. Independent current sources appear in the right hand side of node equations. However, dependent current sources affect the coefficients of node voltage variables in the left-hand side of node equations.

The controlling variable of a linear dependent current source will be a voltage or current existing elsewhere in the circuit. But, any voltage or current variable in the circuit can be expressed in terms of node voltage variables. Hence, the dependent current source function can be expressed in terms of node voltage variables. Therefore, dependent current sources will affect the coefficients of node equation, *i.e.*, they will change the nodal conductance matrix. We will see that they destroy the symmetry of the nodal conductance matrix.

We develop the nodal analysis procedure for this kind of circuits through two examples. The first example has node voltages that are not constrained by independent voltage sources and the second one has node voltage variables constrained by independent voltage source.

Example : 4.5-1

Solve the circuit (a) in Fig. 4.5-1 completely.

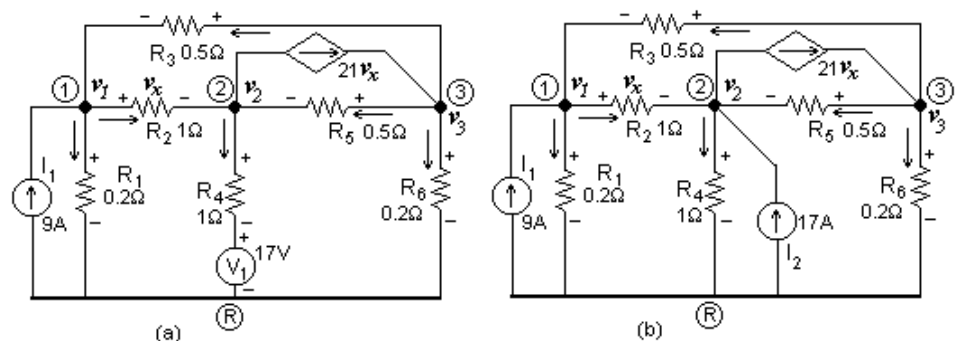


Fig. 4.5-1 (a) Circuit for Example : 4.5-1 (b) Circuit after Node Reduction by Source Transformation

Solution

Step-1: Look for independent voltage sources in series with resistors and apply source transformation on such combinations.

There is one such combination in this circuit. It is V_1 in series with R_4 . Applying source transformation on this combination results in an independent current source of 17 A in parallel with R_4 as shown in circuit (b) of Fig. 4.5-1.

Step-2: Assign node voltage variables at those nodes where the node voltage variable is not decided directly by an independent voltage source or indirectly by already assigned node voltage variables and independent voltage source functions.

Now all the three non-reference nodes in circuit (b) are unconstrained nodes and hence we assign three node voltage variables v_1 , v_2 and v_3 as shown in the figure.

Step-3: Identify the controlling variables of dependent current sources in terms of the node voltage variables assigned in the last step and rewrite the source functions of dependent sources in terms of node voltage variables.

v_x is the controlling variable in this circuit. But v_x is the voltage across R_2 and $= v_1 - v_2$. Therefore the current source function is $k(v_1 - v_2)$ with $k = 21$.

Step-4: Prepare the node equations for the reduced circuit and solve them for node voltage variables.

The node equations are listed below.

$$\text{Node -1} \quad G_1 v_1 + G_2 (v_1 - v_2) + G_3 (v_1 - v_3) = I_1$$

$$\text{Node -2} \quad G_4 v_2 + G_2 (v_2 - v_1) + G_5 (v_2 - v_3) + k(v_1 - v_2) = I_2$$

$$\text{Node -3} \quad G_6 v_3 + G_3 (v_3 - v_1) + G_5 (v_3 - v_2) - k(v_1 - v_2) = 0$$

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Casting these equations in matrix form,

$$\begin{bmatrix} (G_1 + G_2 + G_3) & -G_2 & -G_3 \\ -G_2 + k & (G_2 + G_4 + G_5 - k) & -G_5 \\ -G_3 - k & -G_5 + k & (G_3 + G_5 + G_6) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & G_4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ V_1 \end{bmatrix} \quad (4.5-1)$$

Eqn. 4.5-1 is in the form $YV = CI$ where Y is the nodal conductance matrix. However, the nodal conductance matrix is now *asymmetric* and can not be written down easily by inspection. However, the equation confirms that all node voltages (and hence all element voltages and currents) can be expressed as a linear combination of independent source functions.

Substituting the numerical values,

$$\begin{bmatrix} 8 & -1 & -2 \\ 20 & -17 & -2 \\ -23 & 19 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ 17 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 17 \\ 0 \end{bmatrix}$$

Solving for the voltage vector by Cramer's rule, $v_1 = 2$ volts, $v_2 = 1$ volt and $v_3 = 3$ volts.

Step-5: Use these node voltage values in the original circuit to obtain element voltages and currents.

Now the voltage across elements and current through them can be obtained by inspection. The complete solution is shown in Fig. 4.5-2.

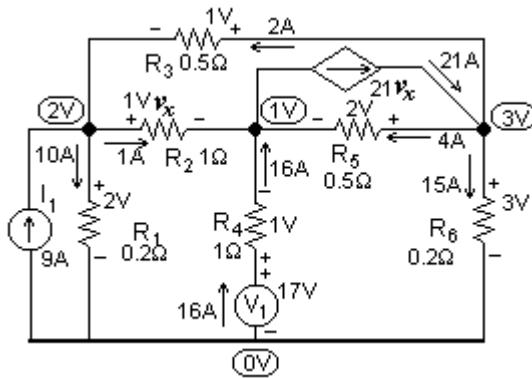


Fig. 4.5-2 Complete Circuit Solution in Example : 4.5-1

Example : 4.5-2

Solve the circuit (a) in Fig. 4.5-3 completely.

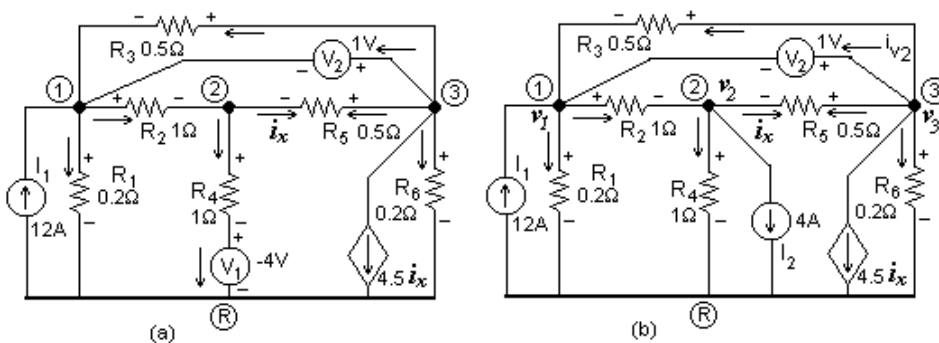


Fig. 4.5-3 (a) Circuit for Example : 4.5-2
(b) Circuit after Node Reduction by Source Transformation

Solution

Step-1: Look for independent voltage sources in series with resistors and apply source transformation on such combinations.

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There is one such combination in this circuit. It is V_1 in series with R_4 . Applying source transformation on this combination results in an independent current source of 4A in parallel with R_4 as shown in circuit (b) of Fig. 4.5-3.

Step-2: Assign node voltage variables at those nodes where the node voltage variable is not decided directly by an independent voltage source or indirectly by already assigned node voltage variables and independent voltage source functions.

We start at left-most node of the circuit (b) and assign a node voltage variable v_1 there since that node is not directly constrained by a voltage source. Moving to node-2, we see that the node voltage at that node can not be obtained from the already assigned variable v_1 and that there is no direct constraint at that node. Hence we assign a node voltage variable v_2 at that node. Now, the node voltage at node-3 can be obtained as $v_1 + V_2$ and a node voltage variable is not needed at that node. Therefore, there are only two node voltage variables in this circuit.

Step-3: Identify the controlling variables of dependent current sources in terms of the node voltage variables assigned in the last step and rewrite the source functions of dependent sources in terms of node voltage variables.

i_x is the controlling variable in this circuit. But $i_x = G_5 [v_2 - (v_1 + V_2)]$. Therefore the current source function is $kG_5[v_2 - (v_1 + V_2)]$, with $k = 4.5$.

Step-4: Prepare the node equations for the reduced circuit and solve them for node voltage variables. Ignore node equation at nodes where voltage sources are connected directly to reference node. Combine the node equations at the end nodes of voltage sources connected between two non-reference nodes.

The node equations are listed below.

$$\text{Node -1} \quad G_1 v_1 + G_2 (v_1 - v_2) - G_3 V_2 - i_{V_2} = I_1$$

$$\text{Node -2} \quad G_4 v_2 + G_2 (v_2 - v_1) + G_5 (v_2 - v_1 - V_2) = -I_2 = G_4 V_1$$

$$\text{Node -3} \quad G_6 (v_1 + V_2) + G_3 V_2 + G_5 (v_1 + V_2 - v_2) + kG_5 (v_2 - v_1 - V_2) + i_{V_2} = 0$$

Combining the node equations at node-1 and node-3 to eliminate i_{V_2} ,

$$\text{Node -1} + \text{Node -3} \quad G_1 v_1 + G_2 (v_1 - v_2) + G_6 (v_1 + V_2) + G_5 (v_1 + V_2 - v_2) + kG_5 (v_2 - v_1 - V_2) = I_1$$

$$\text{Node -2} \quad G_4 v_2 + G_2 (v_2 - v_1) + G_5 (v_2 - v_1 - V_2) = -I_2 = G_4 V_1$$

Casting these equations in matrix form,

$$\begin{bmatrix} (G_1 + G_2 + G_6 + (1-k)G_5) & -(G_2 + (1-k)G_5) \\ -(G_2 + G_5) & (G_2 + G_4 + G_5) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -(G_6 + (1-k)G_5) \\ 0 & G_4 & G_5 \end{bmatrix} \begin{bmatrix} I_1 \\ V_1 \\ V_2 \end{bmatrix}$$

This equation is in the form $\mathbf{YV} = \mathbf{CU}$ where \mathbf{Y} is the nodal conductance matrix. But, the nodal conductance matrix is now *asymmetric* and can not be written down easily by inspection. However, the equation confirms that all node voltages (and hence all element voltages and currents) can be expressed as a linear combination of independent source functions.

Substituting the numerical values,

$$\begin{bmatrix} 4 & 6 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 12 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ -2 \end{bmatrix}$$

Solving for the voltage vector by Cramer's rule, $v_1 = 2$ volts, $v_2 = 1$ volt and $v_3 = v_1 + V_2 = 3$ volts.

Step-5: Use these node voltage values in the original circuit to obtain element voltages and currents for resistors and current sources.

The voltage across resistive elements and current sources and currents through resistive elements can be obtained by inspection. The currents through independent voltage sources in series with resistors can also be obtained at this stage.

Step-6: Use appropriate node equations to solve for currents through the remaining independent voltage sources.

The current through the independent voltage source V_2 has to be determined. We use the node equation at node-1 for this purpose.

and v_2 . Hence we assign a node voltage variable v_3 at that node. Therefore, there are three node voltage variables in this circuit.

Step-3: Identify the controlling variables of dependent current sources in terms of the node voltage variables assigned in the last step and rewrite the source functions of dependent sources in terms of node voltage variables.

i_x is the controlling variable for the dependent current source at node-1 in the circuit (b) of Fig. 4.6-1. But $i_x = G_3 [v_1 - v_3]$. Therefore, the current source function is $k_1 G_1 G_3 [v_1 - v_3]$ with $k_1 = -0.9$.

v_y is the controlling variable for the dependent current source at node-3. But $v_y = v_2$. Therefore, the current source function at node-3 is $k_2 v_2$ with $k_2 = 21$.

Step-4: Prepare the node equations for the reduced circuit and solve them for node voltage variables. Ignore node equation at nodes where voltage sources are connected directly to reference node. Combine the node equations at the end nodes of voltage sources connected between two non-reference nodes.

The node equations are listed below.

$$\text{Node -1 } G_1 v_1 + G_2 (v_1 - v_2) + G_3 (v_1 - v_3) + k_1 G_1 G_3 (v_1 - v_3) = 0$$

$$\text{Node -2 } G_4 v_2 + G_2 (v_2 - v_1) + G_5 (v_2 - v_3) = -I_1$$

$$\text{Node -3 } G_6 v_3 + G_3 (v_3 - v_1) + G_5 (v_3 - v_2) + k_2 v_2 = 0$$

Substituting the numerical values and casting these equations in matrix form,

$$\begin{bmatrix} 17 & -1 & -11 \\ -1 & 4 & -2 \\ -2 & -23 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} [4]$$

This equation is in the form $YV = CU$ where Y is the nodal conductance matrix. But, the nodal conductance matrix is now *asymmetric* and can not be written down easily by inspection. However, the equation confirms that all node voltages (and hence all element voltages and currents) can be expressed as a linear combination of independent source functions.

Solving for the voltage vector by Cramer's rule, $v_1 = 2 \text{ V}$, $v_2 = 1 \text{ V}$ and $v_3 = 3 \text{ V}$.

Step-5: Use these node voltage values in the original circuit to obtain element voltages and currents for resistors and current sources.

The voltage across resistive elements and current sources and currents through resistive elements can be obtained by inspection. The currents through independent voltage sources in series with resistors can also be obtained at this stage.

The complete solution is marked in Fig. 4.6-2.

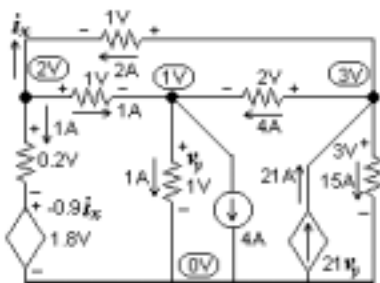


Fig. 4.6-2 Complete Solution for Circuit (a) in Fig. 4.6-1

Example : 4.6-2

Solve the circuit (a) in Example : 4.6-2 by nodal analysis.

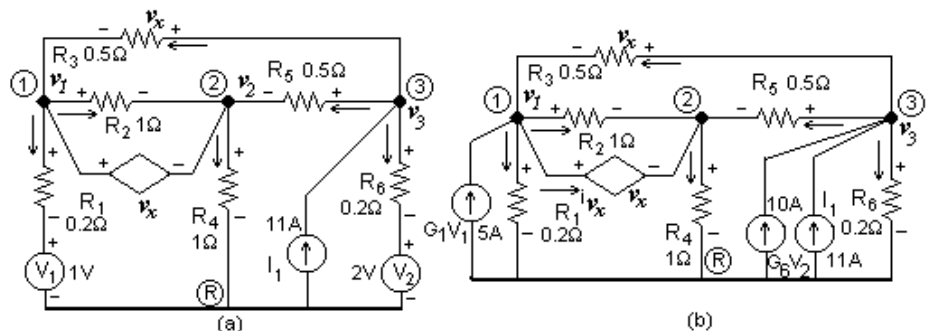


Fig. 4.6-3 (a) Circuit for Example : 4.6-2 (b) Circuit after Node Reduction by Source Transformation

Solution

Step-1: Look for independent voltage sources and dependent voltage sources in series with resistors and apply source transformation on such combinations.

There are two such combinations in this circuit. They are V_1 in series with R_1 and V_2 in series with R_6 . Applying source transformation on these combinations results in circuit (b) of Fig. 4.6-3.

Nodal Analysis Procedure

Step-1: Assign reference current directions and reference polarities for voltages of all elements as per passive sign convention. Look for independent voltage sources and dependent voltage sources in series with resistors and apply source transformation on such combinations to convert them into current sources in parallel with resistors. This is called 'node reduction'. The resulting circuit is referred to as the reduced circuit.

Step-2: Select a reference node. Assign node voltage variables at those nodes where the node voltage variable is not decided directly by a voltage source (independent or dependent source) or indirectly by already assigned node voltage variables and voltage source functions.

Step-3: Identify the controlling variables of dependent current sources in terms of the node voltage variables assigned in the last step and express the source functions of all dependent sources in terms of node voltage variables.

Step-4: Prepare the node equations for the reduced circuit. Ignore node equation at nodes where voltage sources (independent or dependent sources) are connected directly to reference node. Combine (add) the node equations at the end nodes of voltage sources (independent or dependent sources) connected between two non-reference nodes. The number of equations at the end of this step will be equal to number of nodes minus number of irreducible independent voltage sources.

Step-5: Solve for node voltage variables by elimination technique. The equations may also be expressed as a matrix equation and solved by using Cramer's rule or matrix inversion.

Step-6: Use these node voltage values in the original circuit to obtain element voltages and currents for resistors and voltage across current sources. This is done by applying KVL in various loops in the circuit along with Ohm's law for linear resistors.

Step-7: Use appropriate node equations to solve for currents through the independent and dependent voltage sources.

Example : 4.6-3

Solve the circuit in Fig. 4.6-5 by nodal analysis.

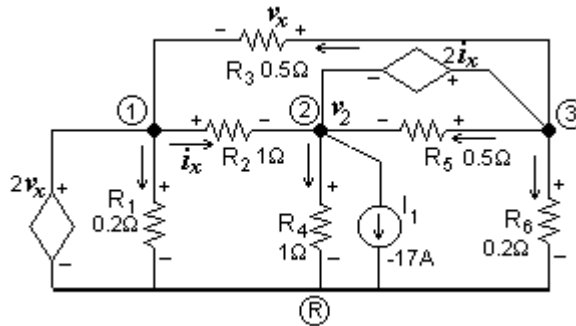


Fig. 4.6-5 Circuit for Nodal Analysis in Example : 4.6-3

Solution

Node-1 is constrained by the dependent source to reference node and hence no node voltage variable can be assigned there. Node-2 is assigned a node voltage variable v_2 . Then, the node voltage variable gets fixed as $v_2 + 2i_x$ through the dependent voltage source connected between node-2 and node-3. Thus, this circuit has only one node voltage variable to be solved for.

Node equation at node-1 is not needed for determining node voltages. However, it will be needed later for determining the current through the dependent voltage source connected at that node.

$$v_x = v_2 + 2i_x - 2v_x \text{ and } i_x = 2v_x - v_2$$

Solving these two equations we get, $v_x = v_2$ and $i_x = v_2$.

The KCL equations at node-2 and node-3 can be combined to form a single equation in v_2 .

$$\text{This combined equation will be } v_2 + (v_2 - 2v_x) + 5(v_2 + 2i_x) + 2(v_2 + 2i_x - 2v_x) = 17.$$

Substituting for v_x and i_x in terms of v_2 , we get, $17v_2 = 17 \Rightarrow v_2 = 1$ volt.

Now, $v_x = v_2 = 1$ volt and $i_x = 1$ A. Therefore the node voltages are 2 volts, 1 volt and 3 volts respectively at node-1, node-2 and node-3. Then the node equation at node-1 can be employed to find current into the positive terminal of dependent source as -9 A. Node equation at node-3 is used to determine the current into the positive terminal of second dependent source as -21 A. The complete solution is marked in Fig. 4.6-6.

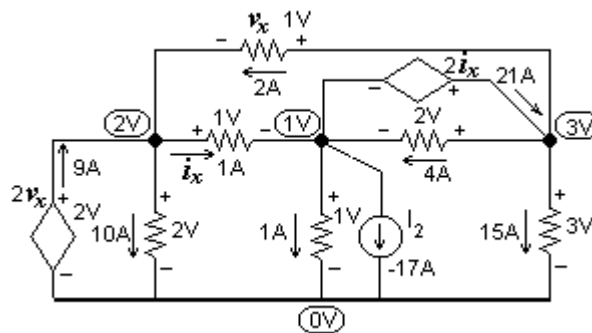


Fig. 4.6-6 Solution for Circuit in Example : 4.6-3

We have completed the development nodal analysis technique for memoryless circuits containing linear resistors, four kinds of linear dependent sources, independent voltage sources and independent current sources.

The *general nodal analysis procedure* that has emerged is summarized in the left side-box.