$$(20+R)^2-(10+R)^2=499.61, i.e., (30+2R)\times 10=499.61 \Rightarrow R=9.98\Omega \approx 10\Omega .$$

Using the value of R = 10 Ω in $(10+R)^2+X^2=500.25$ we get X =10.01 Ω ≈10 Ω.

Thus the Thevenin's equivalent of the circuit is 100∠0^0 V rms in series with (10 + j10) Ω.

(i) If a reactance can be put in series with the load resistance and the value of reactance and resistance can be independently adjusted, then, maximum power transfer will take place under *conjugate impedance matching* condition. Therefore R_L must be 10 Ω and X_L must be –*j*10 Ω for maximum average power transfer to load. The maximum power transferred under this condition will be = $R(V_{oc}/2R)^2 = V_{oc}^2/4R$ watts where V_{oc} is the rms open-circuit voltage. Therefore, the power transferred to (10-*j*10) Ω is 250 watts.

(ii) The condition for maximum power transfer has to be derived for this case. Let R_L be the load resistance. Then the current phasor = $\dfrac{V_{oc}}{(R+R_L)+jX}$ and rms value (*i.e.*, magnitude of the phasor, assuming V_{oc} is the rms open-circuit voltage) of current is $=\dfrac{V_{oc}}{\sqrt{(R+R_L)^2+X^2}}$. Power in R_L = $R_L \times I_{rms}{}^2$ = $\dfrac{R_L V_{oc}{}^2}{(R+R_L)^2+X^2}$. Value of R_L for maximizing this quantity is found by equating its derivative with respect to R_L to zero.

$$\frac{dP_L}{dR_L}=0 \Rightarrow (R+R_L)^2+X^2-2R_L(R+R_L)=0 \Rightarrow R_L=\sqrt{R^2+X^2} .$$

Thus, maximum power transfer takes place in a pure resistive load when load resistance is equal to magnitude of Thevenin's equivalent impedance of the power delivery circuit. The required load resistance in this example is $\sqrt{10^2+10^2}=14.14\Omega$ and the power transferred is = $\dfrac{100^2}{(10+14.14)^2+10^2}\times 14.14=207\,\text{watts}$.

8.8 Phasor Diagrams

We have understood a *phasor* as a *complex amplitude* of a complex exponential function that varies in time as per $e^{j\omega t}$ till now. We lend a little more color to *phasors* in this section. We are motivated by uniform circular motion that is a part of school Physics. We take up a time-domain signal $v_S(t) = V_m \cos\omega t\, u(t)$ represented by a phasor $\boldsymbol{V_S} = V_m \angle 0^0$ and arrive at a *geometric interpretation* for the phasor.

Concept No.1 – Consider a line of length V_m with an arrow at the end (instead of a stone at the end of a taut string) rotating at a constant angular velocity of ω rad/sec in the *counter-clockwise* direction. Let the coordinates of arrow-tip be represented as $x(t)$ and $y(t)$ in the horizontal and vertical directions in a right-handed Cartesian coordinate system as shown in (a) of Fig. 8.8-1.

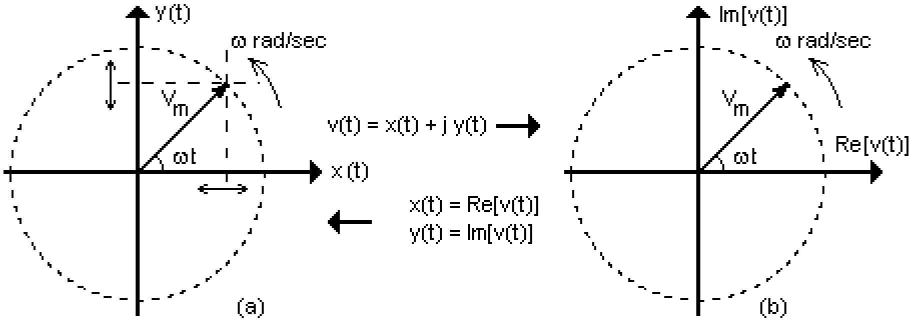


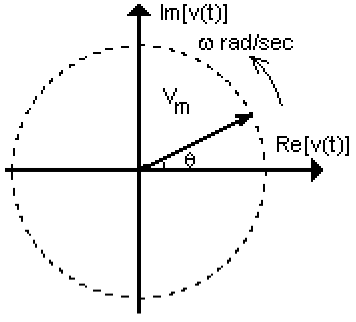
Fig. 8.8-1 (a) A rotating line of length V_m in x-y coordinate system (b) A time-varying complex number in complex plane representing a complex signal constructed using coordinates of arrow-tip in (a)

Assume that the line was collinear with x-axis at t = 0 and then started rotating at ω rad/sec in the direction shown. Then, the angular position of the line in space is given by ωt radians measured in counter-clockwise direction from positive x-axis. The projection of the arrow-tip on the x-axis will then be $V_m\cos\omega t$ and the projection of the arrow-tip on the y-axis will be $V_m\sin\omega t$. Therefore, the signal we started with can be given a *geometric interpretation* of *horizontal projection of arrow-tip of a line of length* V_m *rotating in counter-clockwise direction with a constant angular velocity of* ω *rad/sec, starting from x-axis position at* $t = 0$.

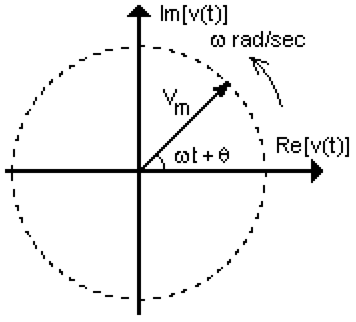
Concept No.2 – Projections on both axes are functions of time. We define a composite function by using these two projection functions. We define a *complex function of time* $v(t) = [x(t) + j\,y(t)]\,u(t)$ by treating the horizontal projection as the *real part* of a complex number and the vertical projection as the *imaginary part* of the same complex number. This complex number can be represented as a point in a *complex plane.* As the line progresses in its rotation, the value of complex number, constructed as explained, too will change. Therefore, the point representing this number in the complex plane also will change with time. A complex number can be geometrically represented by a line with one end at origin and with an arrow at the other end in the complex plane. When the complex number changes with time, the arrow-tip of line representing the number in complex plane will trace out a path in that plane. It must be evident in this case that when the rotating line moves in (a), the corresponding path traced out by the complex number in the complex plane in (b) will also be a circle of radius V_m and the arrow-tip will traverse this path with a constant angular velocity of ω rad/sec.

Thus, a complex signal $v(t) = V_m(\cos\omega t + j\sin\omega t)u(t) = V_m e^{j\omega t}u(t)$ *constructed from coordinates of arrow-tip of a uniformly rotating line in space will be represented geometrically in the 'complex signal plane' by a directed line of length* V_m *rotating uniformly, starting from real axis, in the plane. The values read on the axes of 'complex signal plane' at any instant t are the real and imaginary components of the complex number representing the signal value at that instant.*

Concept No.3 – Let $v_S(t) = V_m \cos(\omega t + \theta)\, u(t)$. Now the arrow-tip of rotating line in (a) of Fig. 8.8-1 will start at $(V_m\cos\theta, V_m\sin\theta)$ *i.e.*, at an angular position of θ at $t = 0$ and will start rotating at ω rad/sec from there. Therefore its angular position at $t = t$ will be $(\omega t + \theta)$. The corresponding complex signal $V_m e^{j(\omega t+\theta)}u(t)$ positions in complex signal plane are marked in Fig. 8.8-2 for $t = 0$ and $t = t$.



(a) At t = 0



(b) At t = t

Fig. 8.8-2 Signal Positions for $V_m e^{j(\omega t+\theta)}$
(a) At $t = 0$ (b) At $t = t$

Concept No.4 – Consider two signals $v_S(t) = V_m\cos(\omega t + \theta_v)$ and $i_S(t) = I_m\cos(\omega t + \theta_i)$ with *same angular frequency*. Their representations in complex signal plane at $t = 0$ and at $t = t$ are shown in (a) and (b) of Fig. 8.8-3.

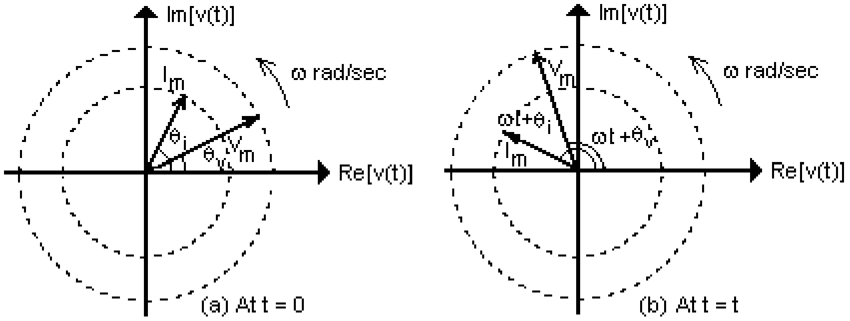


Fig. 8.8-3 Signal Positions of two complex exponential functions at (a) t = 0 (b) t = t

The directed lines of different lengths do change their angular positions with time; but they maintain a constant angular difference at all t. This constant angular difference is the value of angular difference they had at $t = 0$.

Therefore, a set a complex exponential signals, all with same angular frequency but with different initial angular positions, will maintain their relative positions with respect to each other as they rotate in counter-clockwise direction with a constant angular velocity of ω *rad/sec. Such a set of signals with same angular frequency form a 'coherent group' and always stays together with their relative positions unchanging.*

Concept No.5 – Thus rotation aspect is common to all signals at the same angular frequency. That is a piece of information that we can supply at any time and does not have to be carried always in the diagram. What is of interest is the relative phase angles between members of a *coherent group* of complex exponential signals. Therefore, we may suppress the rotation of lines representing complex exponential signals in the complex signal plane - *i.e.*, we may *freeze* the lines at their position at t = 0. This '*freezing'* the signal lines at their initial position converts complex time-functions into complex numbers - *i.e.*, *constant-valued* signals in *complex signal plane*. These *constant-valued* signals in *complex signal plane* are our *phasors.*

Thus, going from *phasor* to time-function involves *'unfreezing'* the directed lines in complex signal plane, allowing them to rotate in counter-clockwise direction at a constant angular velocity of ω rad/sec and extracting the horizontal projection of line end-points, *i.e.*, extracting their real parts.

Concept No.6 – *A diagram depicting a group of coherent (i.e., of same angular frequency) complex exponential signals frozen at their initial position is called a phasor diagram. Angles measured in counter-clockwise direction in a phasor diagram are lead angles and angles measured in clockwise direction in a phasor diagram are lag angles*

Phasor diagram shows the magnitude of phasors as the length of directed arrows to some scale and angle of phasors as angles measured from a reference phasor in counter-clockwise direction. The reference phasor is aligned along the horizontal direction. Since only the relative positions of various phasors in a circuit really matter, any one phasor may be taken as reference phasor and the directions of all other phasors may be marked with respect to this reference phasor provided the absolute phase of reference phasor is preserved for later use. That is, a group of directed lines originating from origin may be rotated as a whole to a new position without affecting the relative positions among the members of the group.

Concept No.7 – Multiplying a phasor by j or e^{j90^0} or $1\angle 90^0$ amounts to rotating it by 90^0 in the counter-clockwise direction in the phasor diagram. This amounts to converting a $\cos\omega t$ into $\cos(\omega t + 90^0) = -\sin\omega t$ in time-domain and, hence, is equivalent to introducing a phase lead of 90^0. Multiplying a phasor by $-j$ or e^{-j90^0} or $1\angle\text{-}90^0$ amounts to rotating it by 90^0 in the clockwise direction in the phasor diagram. This amounts to converting a $\cos\omega t$ into $\cos(\omega t - 90^0) = \sin\omega t$ in time-domain and hence is equivalent to introducing a phase lag of 90^0.

Concept No. 8 – Phasors in a phasor diagram can be added and subtracted by employing *parallelogram law of addition of complex numbers in complex plane.* This law is same as the law of addition of vectors in space coordinates.

A special note on Phasors

Phasors are complex amplitudes used to represent sinusoidal quantities under sinusoidal steady-state condition.

There should be only one value of angular frequency in the circuit. All sinusoidal sources must be at the same frequency. Therefore, a phasor diagram can be drawn only for a circuit that is in sinusoidal steady-state under the influence of one or more sinusoidal sources at same angular frequency.

If there are different frequency sinusoids present in the same circuit, different phasor diagrams – one each for each frequency – should be drawn. The phasor solution arrived at from different phasor diagrams will have to be transformed into time-domain quantities using relevant angular frequency values before combining the solutions by invoking Superposition Theorem.

Example : 8.8-1

Draw the phasor diagram showing all voltage phasors and current phasors for a series *RL* circuit with R = 6 Ω, X = 8 Ω at 50 Hz and $v_S(t) = 20\cos 100\pi t$.

Solution

The impedance of the circuit, $\boldsymbol{Z} = (6+j8)\Omega = 10\angle 53.13^0\ \Omega$. Applied voltage phasor $\boldsymbol{V_S} = 20\angle 0^0$ V.The circuit current phasor $\boldsymbol{I} = 20\angle 0^0\ \text{V} \div 10\angle 53.13^0\ \Omega = 2\ \angle\text{-}53.13^0$ A

Voltage phasor across resistor $\boldsymbol{V_R} = 6\ \Omega \times 2\ \angle\text{-}53.13^0\ \text{A} = 12\ \angle\text{-}53.13^0$ V. Voltage phasor across inductor $\boldsymbol{V_L} = j8\ \Omega \times 2\ \angle\text{-}53.13^0\ \text{A} = 16\ \angle 36.87^0$ V

We choose the applied voltage phasor as the reference phasor and align it along horizontal direction. Different scaling for voltage phasor magnitudes and current phasor magnitudes will have to be employed when a phasor diagram shows voltage phasors and current phasors together.

The circuit and phasor diagrams are shown in Fig. 8.8-4.

Note that $\boldsymbol{V_L}$ and $\boldsymbol{V_R}$ add to form $\boldsymbol{V_S}$ by parallelogram law of addition. The inductor voltage is seen to lead the circuit current by 90^0 and current lags the applied voltage by the impedance angle equal to 53.13^0.

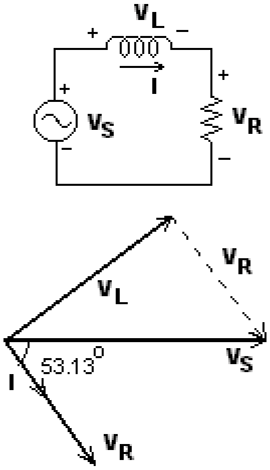


Fig. 8.8-4 An RL Circuit and its Phasor Diagram

Example : 8.8-2

Draw the phasor diagram showing all voltage phasors and current phasors for a series *RC* circuit with $R = 6\ \Omega$, $X = -8\ \Omega$ at 50 Hz and $v_S(t) = 20\cos 100\pi t$.

Solution

The impedance of the circuit, $\boldsymbol{Z} = (6\text{-}j8)\Omega = 10\angle\text{-}53.13^0\ \Omega$. Applied voltage phasor $\boldsymbol{V_S} = 20\angle 0^0$. The circuit current phasor $\boldsymbol{I} = 20\angle 0^0\ \text{V} \div 10\angle\text{-}53.13^0\ \Omega = 2\ \angle 53.13^0$ A. Voltage phasor across resistor $\boldsymbol{V_R} = 6\ \Omega \times 2\ \angle\text{-}53.13^0$ A $= 12\ \angle 53.13^0$ V. Voltage phasor across inductor $\boldsymbol{V_L} = \text{-}j8\ \Omega \times 2\ \angle 53.13^0$ A $=16\ \angle\text{-}36.87^0$ V. We choose the applied voltage phasor as the reference phasor and align it along horizontal direction. The circuit and phasor diagrams are shown in Fig. 8.8-5.

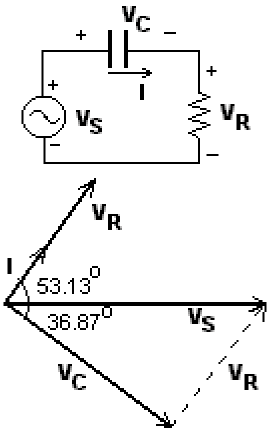


Fig. 8.8-5 The RC Circuit and its Phasor Diagram

Example : 8.8-3

A series *RLC* circuit is excited by a voltage source $v_S(t) = 100\cos\omega t\ u(t)$ volts. The inductive reactance at ω is 10Ω and capacitive reactance at ω is $10\ \Omega$. The resistor has 1Ω value. Draw the phasor diagram of the circuit under sinusoidal steady-state condition.

Solution

The impedance of the circuit at ω rad/sec $\boldsymbol{Z} = 10+j10\text{-}j10 = 10\angle 0^0\ \Omega$

Applied voltage phasor, $\boldsymbol{V_S} = 100\angle 0^0$ volts

$\therefore$ Circuit current phasor, $\boldsymbol{I} = 100\angle 0^0$ amps

$\therefore$ Voltage phasor across *R*, $\boldsymbol{V_R} = 100\angle 0^0$ volts

$\therefore$ Voltage phasor across *L*, $\boldsymbol{V_L} = j10\ \Omega \times 100\angle 0^0$ amps $= 1000\angle 90^0$ volts

$\therefore$ Voltage phasor across *C*, $\boldsymbol{V_L} = \text{-}j10\ \Omega \times 100\angle 0^0$ amps $= 1000\angle\text{-}90^0$ volts

Note that the voltage across capacitor and inductor are in phase opposition. Therefore, they cancel each other completely, thereby leaving the entire supply voltage to the resistor. Hence, the current under this condition is the maximum current that the circuit can have with given amplitude of applied voltage. This happens because the impedance of capacitor and inductor are equal in magnitude and opposite in sign. They cancel out, making the impedance of the circuit a minimum of *R* at this frequency. This is the *resonance* condition in a series *RLC* circuit.

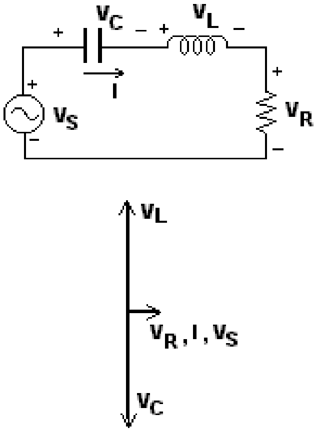


Fig. 8.8-6 A RLC Circuit and its Phasor Diagram

Example : 8.8-4

A sinusoidal current source $i_S(t) = 5\cos(100\pi t - 45^0)$ amps is applied across a parallel combination of an inductor , a resistor and a capacitor. Find the steady-state currents in elements and voltage across the combination using phasor diagram. The resistance value is $6\ \Omega$ and the values of reactance of inductor and capacitor at $\omega = 100\pi$ rad/sec are $8\ \Omega$ and $-4\ \Omega$ respectively.

Solution

This example calls for use of phasor diagram to solve the circuit under sinusoidal steady-state. Hence the phasor quantities are unknown when we draw the phasor diagram. In this kind of a situation, the phasor that we choose as reference phasor has to have the property that all the other phasors in the circuit can be worked out from this phasor employing KCL, KVL and element relationship. Applied voltage or current will not be suitable for this purpose. It has to be one of the response variables. But, if it is one of the response variables, its magnitude will be unknown and hence we can not fix the scale in phasor diagram. Thus, the phasor diagram is drawn by assigning an arbitrary, but known, length to the phasor that is chosen as the reference phasor. The scale in the diagram will emerge from the known applied voltage or current phasor once the diagram is completed according to KCL, KVL and element relationships.

We choose the current in the resistor as the reference phasor in this example. The circuit and phasor diagram are shown in Fig. 8.8-7 .

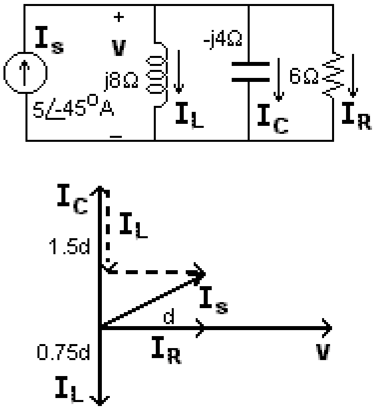


Fig. 8.8-7 Circuit and Phasor Diagram in Example : 8.8-4

We have set $\boldsymbol{I_R}$ in the horizontal direction. This does not mean that $i_R(t)$ will be a $\cos 100\pi t$ wave. Once we solve the phasor diagram completely, the source current phasor will come out with some angle other than its actual angle of -45^0. Then we will rotate the entire phasor diagram such that $\boldsymbol{I_S}$ takes up -45^0 position in the diagram. The other phasors will then take up suitably shifted positions. The new angular positions can be calculated from the known angular position of $\boldsymbol{I_S}$ and apparent angular position of $\boldsymbol{I_S}$ in the diagram shown in Fig. 8.8-7.

Let d be the length that we used to represent $\boldsymbol{I_R}$. Then the current through the capacitor $\boldsymbol{I_C} = R\boldsymbol{I_R}/jX_C$ will have a magnitude of 6/4 = 1.5 times that of $\boldsymbol{I_R}$ and hence needs a line of length $1.5d$ in 90^0 position in the diagram. The current through inductor $\boldsymbol{I_L} = R\boldsymbol{I_R}/jX_L$ will have a magnitude of 6/8 = 0.75 times that of $\boldsymbol{I_R}$ and hence needs a line of length $0.75d$ in -90^0 position in the diagram. Moving a copy of $\boldsymbol{I_L}$ to the tip of $\boldsymbol{I_C}$ takes us to $(\boldsymbol{I_C} + \boldsymbol{I_L})$ and placing a copy of $\boldsymbol{I_R}$ at the tip of $(\boldsymbol{I_C} + \boldsymbol{I_L})$ takes us to the tip of $\boldsymbol{I_S}$ phasor. Thus we complete the diagram. Now, we either measure the length of $\boldsymbol{I_S}$ phasor or calculate it as $\sqrt{(0.75d)^2 + d^2} = 1.25d$ from the geometry of the figure. But this must be equal to 5 amps since the amplitude of $i_S(t)$ is stated to be 5 amps. Therefore a length of d stands for 5/1.25 = 4 amps. Therefore magnitude of $\boldsymbol{I_R}$ is 4 amps, of $\boldsymbol{I_C}$ is 6 amps and $\boldsymbol{I_L}$ is 3 amps.

The angle of $\boldsymbol{I_S}$ phasor in the phasor diagram is $\tan^{-1}(0.75) = 36.9^0$. But we know that actual angle of $\boldsymbol{I_S}$ phasor is -45^0. Therefore angle of $-(45^0+36.9^0) = -81.9^0$ will have to be added to all phasors in the diagram.

$\therefore \boldsymbol{I_S} = 5\angle{-45^0}$ A, $\boldsymbol{I_R} = 4\angle{-81.9^0}$ A, $\boldsymbol{I_C} = 6\angle 8.1^0$ A and $\boldsymbol{I_L} = 3\angle{-171.9^0}$ A and $\boldsymbol{V} = 24\angle{-81.9^0}$ V is the phasor solution of the circuit.

Corresponding time-domain functions are:

$i_S(t) = 5\cos(100\pi t - 45^0)$ amps, $i_R(t) = 4\cos(100\pi t - 81.9^0)$ amps, $i_C(t) = 6\cos(100\pi t + 8.1^0)$ amps, $i_L(t) = 3\cos(100\pi t - 171.9^0)$ amps and $v(t) = 24\cos(100\pi t - 81.9^0)$ volts.

We could have used $\boldsymbol{I_C}$ or $\boldsymbol{I_L}$ or $\boldsymbol{V}$ as the reference phasor and developed the phasor diagram to arrive at the same solution. However, we could not have used $\boldsymbol{I_S}$ as the reference phasor since we wold not have been able to proceed any further with that choice.

Example : 8.8-5

Two impedances $\boldsymbol{Z_1} = 6+j8\ \Omega$ and $\boldsymbol{Z_2} = 8-j6\ \Omega$ are in parallel and the whole combination is in series with a third impedance $\boldsymbol{Z_3} = 5+j5\ \Omega$. The circuit is driven by a sinusoidal voltage source $v_S(t) = 50\sin 100\pi t$. Solve the circuit by phasor diagram method.

Solution

The circuit and phasor diagrams are shown in Fig. 8.8-8.

We choose the current phasor $\boldsymbol{I}$ as the reference phasor.

The impedance values are converted to polar form as $\boldsymbol{Z_1} = 10\angle 53.1^0\ \Omega$, $\boldsymbol{Z_2} = 10\angle{-36.9^0}\ \Omega$ and $\boldsymbol{Z_3} = 7.07\angle 45^0\ \Omega$. The parallel combination, $\boldsymbol{Z_1}//\boldsymbol{Z_2}$ is $= 7+j1\ \Omega = 7.07\angle 8.1^0$.

$$\boldsymbol{I_1} = \frac{\boldsymbol{Z_2}}{\boldsymbol{Z_1}+\boldsymbol{Z_2}}\boldsymbol{I} = (0.5 - j0.5)\boldsymbol{I} = 0.707\angle{-45^0}\,\boldsymbol{I}$$

$$\boldsymbol{I_2} = \frac{\boldsymbol{Z_1}}{\boldsymbol{Z_1}+\boldsymbol{Z_2}}\boldsymbol{I} = (0.5 + j0.5)\boldsymbol{I} = 0.707\angle 45^0\,\boldsymbol{I}$$

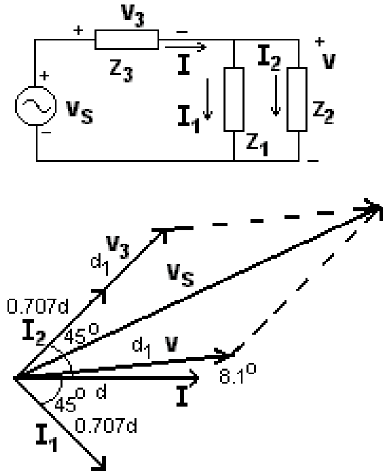


Fig. 8.8-8 Circuit and Phasor Diagram for Example : 8.8-5

We use a length d for $\boldsymbol{I}$. Then, the length to be used for $\boldsymbol{I_1}$ and $\boldsymbol{I_2}$ are $0.707d$ and they are oriented at -45^0 and 45^0 respectively. Next we draw the $\boldsymbol{V} = 7.07\angle 8.1^0\,\boldsymbol{I}$ phasor at 8.1^0 with respect to horizontal and use a convenient length d_1 if the length $7.07d$ is not suitable. The $\boldsymbol{V_3}$ phasor is also of the same length since $\boldsymbol{V_3} = \boldsymbol{Z_3}\,\boldsymbol{I} = 7.07\angle 45^0\,\boldsymbol{I}$. But it is to be drawn at 45^0 position.

The phasors $\boldsymbol{V}$ and $\boldsymbol{V_3}$ on addition as per parallelogram law should result in $\boldsymbol{V_S}$. The length of $\boldsymbol{V_S}$ must be $2 \times d_1 \times \cos[(45^0 - 8.1^0)/2] = 1.9\,d_1$. But this length must stand for 50 V and hence d_1 must stand for 26.3 V. Therefore, magnitudes of $\boldsymbol{V}$ and $\boldsymbol{V_3}$ are 26.3 V. Since $\boldsymbol{I} = \boldsymbol{V_3}/\,\boldsymbol{Z_3}$, magnitude of $\boldsymbol{I}$ will be 3.72 A. Now, magnitudes of $\boldsymbol{I_1}$ and $\boldsymbol{I_2}$ are 0.707 times the magnitude of $\boldsymbol{I}$. Hence they are of 2.63 A magnitude.

The angle of $\boldsymbol{V_S}$ as per the phasor diagram is $8.1^0 + (45^0\text{-}8.1^0)/2 = 26.55^0$. But since $v_S(t) = 50 \sin 100\pi t$, the actual phase angle of $\boldsymbol{V_S}$ is -90^0 with respect standard cosine wave. Therefore, an angle of -116.55^0 has to be added to the angle of all phasors in the phasor diagram shown in Fig. 8.8-8.

Therefore, the sinusoidal steady-state solution of the circuit is obtained as:

$\boldsymbol{V_S} = 50\angle\text{-}90^0$, and,

$\boldsymbol{V} = 26.3\angle\text{-}108.45^0$ V and $\boldsymbol{V_3} = 26.3\ \angle\text{-}71.55^0$ V.

$\boldsymbol{I} = 3.72\ \angle\text{-}116.55^0$ A, $\boldsymbol{I_1} = 2.63\ \angle\text{-}151.55^0$ A, $\boldsymbol{I_2} = 2.63\ \angle\text{-}71.55^0$ A

The time-domain functions are:

$v_S(t) = 50 \cos(100\pi t - 90^0) = 50 \sin 100\pi t$ volts

$v(t) = 26.3 \cos(100\pi t - 108.45^0) = 26.3 \sin(100\pi t - 18.45^0)$ volts

$v_3(t) = 26.3 \cos(100\pi t - 71.55^0) = 26.3 \sin(100\pi t + 18.45^0)$ volts

$i(t) = 3.72 \cos(100\pi t - 116.55^0) = 3.72 \sin(100\pi t - 26.55^0)$ amps

$i_1(t) = 2.63 \cos(100\pi t - 151.55^0) = 2.63 \sin(100\pi t - 61.55^0)$ amps

$i_2(t) = 2.63 \cos(100\pi t - 71.55^0) = 2.63 \sin(100\pi t + 18.45^0)$ amps

Example : 8.8-6

Three sinusoidal voltage sources - $v_1(t)$, $v_2(t)$ and $v_3(t)$ - with angular frequency of 100π rad/sec and amplitudes of 63 V, 52 V and 25 V respectively, are connected in series along with a 10 Ω resistor to form a closed loop. The voltage sources are connected in such a way that they aid each other in the loop. The current in 10 Ω resistor is found to be zero. Find $v_1(t)$, $v_2(t)$ and $v_3(t)$.

Solution

The statement of the problem makes it clear that $v_1(t) + v_2(t) + v_3(t) = 0$. Therefore, the phasor diagram of the three voltage phasors will form a closed triangle. The phasor diagram is shown in Fig. 8.8-9.

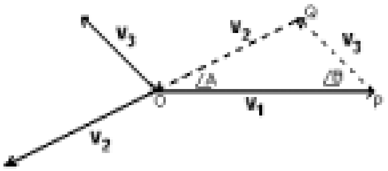


Fig. 8.8-9 Phasor Diagram in Example : 8.8-6

The phasor diagram is drawn as follows. Choose a suitable scale and draw the line OP to represent magnitude of $\boldsymbol{V_1}$. With O as center, draw a circle of radius 52 to scale. Draw another circle of radius 25 to scale with P as its center. Let the two circles intersect at Q. They will intersect; otherwise the three voltages would not have added up to zero. Join QO and PQ. Create a copy of QO and move it to form $\boldsymbol{V_2}$. Similarly, create a copy of PQ and move it in parallel such that the non-arrow end comes to O to form $\boldsymbol{V_3}$.

Now $\angle$A and $\angle$B can be measured from the diagram. Then $\boldsymbol{V_1} = 63\angle 0^0$, $\boldsymbol{V_2} = 52\angle\text{-}(180\text{-}A)^0$ and $\boldsymbol{V_3} = 25\angle(180\text{-}B)^0$.

The angles $\angle$A and $\angle$B can also be calculated by *Law of Cosines*.

$25^2 = 63^2 + 52^2 - 2\times 63\times 52\times \cos A \Rightarrow A = 22.62^0$

$52^2 = 63^2 + 25^2 - 2\times 63\times 25\times \cos B \Rightarrow B = 53.13^0$

$\therefore \boldsymbol{V_1} = 63\angle 0^0$ V, $\boldsymbol{V_2} = 52\angle\text{-}157.38^0$ V and $\boldsymbol{V_3} = 25\angle 126.87^0$ V.

$\therefore v_1(t) = 63\cos(100\pi t)$V, $v_2(t) = 52\cos(100\pi t - 157.4^0)$V and $v_3(t) = 25\cos(100\pi t + 126.9^0)$ V.

8.9 Apparent Power, Active Power, Reactive Power and Power Factor

Consider a sinusoidal voltage source $v(t) = V_m \cos\omega t$ delivering power to a resistive load R. The current in the resistor is $i(t) = I_m \cos\omega t$ where $I_m = V_m/R$

The instantaneous power is $p(t) = V_m I_m \cos^2\omega t = 0.5\ V_m I_m + 0.5\ V_m I_m \cos 2\omega t$ watts. The first term is a constant and the second term produces an average of zero over a cycle. Therefore average power delivered to resistor is $0.5\ V_m I_m = 0.5\ V_m^2/R = 0.5\ I_m^2 R$. The average power can be expressed as $V_{rms} I_{rms}$ in terms of rms values of voltage and current. Thus, a sinusoidal voltage/current is only as effective as a dc voltage /current of magnitude that is only 70.7% of the amplitude of the sinusoid. The presence of the second term – the term that has as much strength as the average power; but is oscillating at twice the supply frequency – indicates this relative inefficiency of sinusoids compared to dc quantities in carrying power to a load. This is the inevitable price that we have to pay for having opted for sinusoidal waveforms. Hence, we do not complain about the inevitable double-frequency power pulsation that has as much amplitude as the average power that is being delivered to the load.

Consider the same voltage source delivering power to the same resistor- but the resistor is in *parallel* with an inductive reactance X at ω rad/sec as shown in Fig. 8.9-1.

Such a load is called a *reactive load*. The load impedance now is given by

$$\boldsymbol{Z} = R \,//\, jX = \frac{jRX}{R+jX} = \left(\frac{X^2}{R^2+X^2}\right)R + j\left(\frac{R^2}{R^2+X^2}\right)X = \frac{RX}{\sqrt{R^2+X^2}}\angle\tan^{-1}\frac{R}{X}$$

The current delivered by the voltage source will be

$$i(t) = \frac{V_m}{R}\cos\omega t + \frac{V_m}{X}\sin\omega t = I_m\cos(\omega t - \theta)$$

where $I_m = \dfrac{V_m\sqrt{R^2+X^2}}{RX}$ and $\theta = \tan^{-1}\dfrac{R}{X}$

The instantaneous power is $p(t) = V_m I_m \cos\omega t\,\cos(\omega t-\theta)$

$p(t) = V_m(I_m\cos\theta)\cos^2\omega t + V_m(I_m\sin\theta)\sin\omega t\cos\omega t$

$p(t) = \{[0.5V_m(I_m\cos\theta)] + [0.5V_m(I_m\cos\theta)]\cos 2\omega t\,]\} + [0.5\,V_m(I_m\sin\theta)]\sin 2\omega t$

The average power is $0.5\,V_m I_m\cos\theta = 0.5\,V_m \times \dfrac{V_m\sqrt{R^2+X^2}}{RX} \times \cos(\tan^{-1}(R/X))$

$= 0.5\,V_m^2/R$. [since $\cos(\tan^{-1}(R/X)) = \dfrac{X}{\sqrt{R^2+X^2}}$.]

Thus $I_m\cos\theta = V_m/R$ is the same as the current drawn by the resistor alone.

Thus, average power is due to the current drawn by resistor and is the same as before. However, the source has to deliver a higher current to deliver the same amount of power now. The first double-frequency power pulsation (*i.e.,* the $\cos 2\omega t$ term in $p(t)$) is the expected double-frequency pulsation when an average power is being delivered. The second double-frequency pulsating power (*i.e.*, the $\sin 2\omega t$ term) is solely due to the inductor in parallel with resistor - *i.e.,* due to the *reactive* nature of load) and has an amplitude of $0.5\,V_m I_m\sin\theta = 0.5V_m^2/X$. (since $\sin\theta = \cos(\tan^{-1}(X/R)) = \dfrac{R}{\sqrt{R^2+X^2}}$). The presence of the second pulsating power term with non-zero amplitude is an indicator to the fact that the magnitude of current is more than the minimum magnitude of current required to pass on the average power to the load. The minimum current that is required in the circuit to deliver the average power it is delivering now is only $\cos\theta$ times the present current.

If the voltage in a dc circuit is same as V_{rms} of this sinusoidal voltage source and the current in the dc circuit is same as the I_{rms} in this ac circuit, then, the dc source would have delivered $V_{rms}I_{rms}$ watts of average power to the load. Compared to that, the ac circuit delivers only $\cos\theta$ times this power. Thus, effectiveness of utilisation of voltage and current in a reactive circuit under sinusoidal steady-state is compromised by the factor $\cos\theta$ compared to a dc circuit carrying similar voltage and current. This observation leads to a definition of *apparent power* in an ac circuit.

Since the average power in an ac circuit can be different by a factor $\cos\theta$, where θ is the angle between voltage phasor and current phasor, the unit of *watts* is reserved for average power and a unit of *volt-amperes* (*VA*) is assigned to apparent power. Since only the average power contained in the apparent power is *active* in generating useful output from the circuit, average power is called *active power*. The ratio between the active power and apparent power is called the *power factor* of the circuit.

Note that the definitions of apparent power, active power and power factor are applicable for any general periodic waveform context. But the expressions, $V_{rms}I_{rms}\cos\theta$ *for active power and* $\cos\theta$ *for power factor, are applicable only under sinusoidal steady-state condition.*

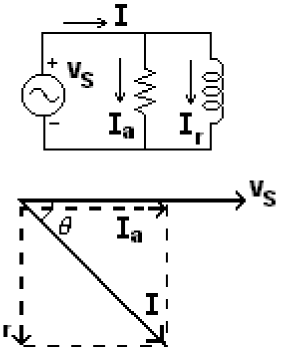


Fig. 8.9-1 A Parallel RL Load and its Phasor Diagram

Apparent power carried by a sinusoidal voltage of rms value V_{rms} and a sinusoidal current of rms value I_{rms} is defined as the actual power that will be carried by a dc voltage of same effective value and a dc current of same effective value -*i.e.,* Apparent Power = $V_{rms} \times I_{rms}$.

Apparent Power = $V_{rms}I_{rms}$

Active Power, $P = V_{rms}I_{rms}\cos\theta$

where θ is the angle by which the voltage phasor leads the current phasor.

Power Factor = $\dfrac{\text{Active Power}}{\text{Apparent Power}}$

$= \cos\theta$

Active and Reactive Components of Current Phasor

$I_m \cos\theta$ is the amplitude of $\cos\omega t$ term in current and $I_m \sin\theta$ is the amplitude of $\sin\omega t$ term in current. $\cos\omega t$ and $\sin\omega t$ terms are represented by phasors that have 90^0 between them. They are called *quadrature components* for this reason. Thus, $I_m \cos\theta$ is the *in-phase* component in current phasor and $I_m \sin\theta$ is the *quadrature* component in current phasor with respect to the voltage phasor. $I_m \cos\theta$, the in-phase component, carries the average power (along with an unavoidable double-frequency pulsating power of equal amplitude), and, $I_m \sin\theta$, the quadrature component, produces a pure double-frequency pulsating power term with zero average content. This pulsating power term is avoidable by making $\theta = 0$ - *i.e.*, by making the load purely resistive.

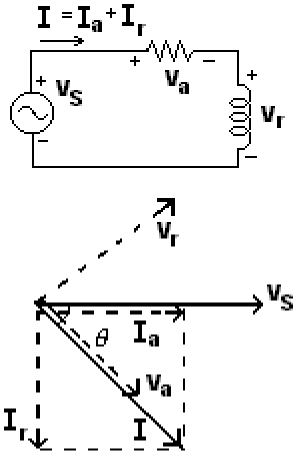


Fig. 8.9-2 A Series RL Load and its Phasor Diagram

<u>Sign of active and reactive components of current</u>

$\boldsymbol{I_a}$ is in phase with voltage phasor by definition and hence its phase is known. Therefore it is common practice to refer to $\boldsymbol{I_a}$ as if it is real number with positive or negative sign.

In the case of $\boldsymbol{I_r}$, it can be specified as a real number with positive or negative sign. Or else, one may add the qualifiers '*capacitive*' or '*inductive*' and then skip the sign.

The *j* in $\boldsymbol{I_r}$ is taken to be implied by the word '*reactive*' in 'reactive component'. Thus, if $V_S = V_m\angle 0^0$ and $\boldsymbol{I} = I_m\angle -\theta$, then, *active component of current* is $I_m\cos\theta$ ($I_{rms}\cos\theta$ A rms) and *reactive component of current* is $-I_m\sin\theta$ ($-I_{rms}\sin\theta$ A rms). If the negative sign is skipped, then we have to state that *inductive component of current* is $I_m\sin\theta$ ($I_{rms}\sin\theta$ A rms).

For instance, a 5A *reactive current* implies 5A of capacitive current, a -5A *reactive current* implies 5A of inductive current. A 5A *inductive current* implies 5A of inductive current and a –5A *inductive current* implies 5A of capacitive current. Similarly, a 5A *capacitive current* implies 5A of capacitive current and *–5A capacitive current* implies 5A of inductive current.

Inductive current component *lags* voltage phasor by 90^0 and capacitive current component *leads* voltage phasor by 90^0.

Any current phasor can be resolved into two components – one in the direction of voltage phasor and one in a direction perpendicular to the voltage phasor. The component in the direction of voltage phasor is the *in-phase* component and this component will carry *active power*. Therefore this component is called *active component of current* and is denoted by a phasor $\boldsymbol{I_a}$. The component in the perpendicular direction to voltage phasor is the *quadrature* component and this component *will not* carry *any average power*. This component is decided by the reactance in the circuit and goes to zero when the circuit is purely resistive. Hence this component is called the *reactive component of current* and is denoted by a phasor $\boldsymbol{I_r}$. The current phasor $\boldsymbol{I}$ is the phasor sum of $\boldsymbol{I_a}$ and $\boldsymbol{I_r}$. This is shown in the phasor diagram in Fig. 8.9-1.

Note that the definition of *active and reactive components of a current phasor* is based on projecting the current phasor along and perpendicular to voltage phasor and is applicable to any current phasor. In the circuit we considered in Fig. 8.9-1, the active component of $\boldsymbol{I}$ could be identified as the current in R and the reactive component could be identified as the current in jX. However, such an identification of active and reactive current components of a given current phasor is not a precondition for their definition. Consider a series *RL* load and its phasor diagram in Fig. 8.9-2.

The current phasor $\boldsymbol{I}$ can be resolved into in-phase component $\boldsymbol{I_a}$ and quadrature component $\boldsymbol{I_r}$ with respect to the voltage phasor as shown in the phasor diagram in Fig. 8.9-2. However, we can not identify these components as real currents flowing in any element since there is only one current in a series circuit and that is $\boldsymbol{I}$. Further, we observe that the voltage phasor also can be resolved into two components –one along the current phasor direction and one in a perpendicular direction. These are the *active component* and *reactive component* of voltage phasor.

We can always think of a parallel-connected resistance and reactance that has same impedance as that of a series connected resistance and reactance at a particular frequency. Then, we can identify the active component of $\boldsymbol{I}$ in a series circuit as the current that will flow in the resistor of a parallel circuit that has same impedance as that of the series circuit. Similarly, we can identify the reactive component of $\boldsymbol{I}$ in a series circuit as the current that will flow in the reactance of a parallel circuit that has same impedance as that of the series circuit. Hence, though resolving voltage phasor along current phasor and resolving current phasor along voltage phasor have the same effect in power equations, we choose to use the quadrature components of current phasor rather than of voltage phasor in subsequent discussion.

The side-note describes the sign convention adopted in specifying active and reactive current components.

Reactive Power and the Power Triangle

We restate the concepts we developed in the last sub-section before we continue with the concept of *reactive power* that tends to be a confusing one to beginners in Electrical Engineering.

- *Let* $v(t) = V_m \cos\omega t$ *volts and* $i(t) = I_m \cos(\omega t - \theta)$ *amps be the steady-state voltage and current at a pair of load terminals as per passive sign convention. Then 'apparent power' which is the power that a dc circuit with same effective values of voltage and current would have delivered is* $V_{rms} I_{rms} = 0.5\ V_m I_m$ *VA. The average* *power which is also called 'active power' is* $P = V_{rms} I_{rms} \cos\theta = 0.5\, V_m I_m \cos\theta$. *The power factor of the circuit which is defined as the ratio of active power to apparent power is* $\cos\theta$.

- *The minimum magnitude of current required to deliver a given amount of power P is given by* $I_{rms} = P/V_{rms}$ *with* $\cos\theta = 1$. *This happens when the driving-point impedance of the load at* ω *rad/sec is effectively a resistance. Then the circuit has* $\theta = 0$, *and, draws power at unity power factor with minimum magnitude of current.*
- *Reactive component in the driving-point impedance of the load circuit makes* θ *non-zero and increases the current magnitude for a given amount of active power. Thus current is under-utilized as far as active power delivery is considered. The power factor of the circuit will be less than unity.*
- *The current phasor can be resolved into 'active component' (=* $I_{rms} \cos\theta$ *A rms) and 'reactive component' (=* $-I_{rms} \sin\theta$ *A rms) by finding its projection along voltage phasor and along a perpendicular to voltage phasor respectively. The 'active current component' may be thought of as the current drawn by the resistor in a parallel connected resistance -reactance combination that has same impedance as that of the load circuit. The 'reactive component of current' is the current drawn by the reactance in that equivalent parallel circuit. The 'active current component' carries the entire 'active power'.*

The utilization of load current in its role as a vehicle to carry active power can be judged from the relative proportion of its active and reactive current components. It can be shown easily that $I_{rms} = \sqrt{I_{a,rms}^2 + I_{r,rms}^2}$ and $I_m = \sqrt{I_{am}^2 + I_{rm}^2}$ where $I_{a,rms}$ and $I_{r,rms}$ indicate the rms values of the components whereas I_{am} and I_{rm} indicate their amplitudes. The following relations also hold between various quantities.

$$\cos\theta = \text{power factor} = \frac{I_{a,rms}}{I_{rms}} = \frac{I_{am}}{I_m}$$

$$\sin\theta = -\frac{I_{r,rms}}{I_{rms}} = -\frac{I_{rm}}{I_m}$$

$$\tan\theta = -\frac{I_{r,rms}}{I_{a,rms}} = -\frac{I_{rm}}{I_{am}}$$

To state that there is some reactive power flow into a load is a disguised way of stating that:

(i) The load impedance has a reactive component.

(ii) The load current has a reactive component that reduces the efficacy of current in carrying active power.

(iii) Therefore, the current magnitude is more than the minimum magnitude needed that is commensurate with actual power transfer taking place.

(iv) Hence, circuit is operating at a power factor less than unity.

$I_{r,rms}$ *and* I_{rm} *contain the sign of reactive component.* Therefore, while power factor is always positive, $\sin\theta$ and $\tan\theta$ are positive for a lagging load and negative for a leading load. *Note that* θ *is the angle by which voltage phasor leads the current phasor.* Power factor of a load is independent of sign of θ and hence a qualifier *lag* or *lead* is to be appended to the number representing power factor to distinguish between positive and negative values of θ. Thus, if $\theta = 45^0$ the power factor is '0.7 lag' and if θ is -45^0 the power factor is '0.7 lead'.

Another method to describe the utilization of load current in carrying active power will be to compare the active and reactive rms components of current after scaling the active rms current component by $+V_{rms}$ and reactive rms current component by $-V_{rms}$. But then, if the *active component of current* is multiplied by $+V_{rms}$, the result is *active power*. Then, it will be tempting to call the product of the *reactive component of current* and $-V_{rms}$ a power – but not a real average power, since this component of current produces only $V_{rms} I_{r,rms} \sin 2\omega t$ term in instantaneous power. Electrical Engineers yielded to this temptation long back and they called it *reactive power* in contrast to *active power*.

Thus reactive power is not a power at all; it is only a power-like measure of reactive component of current. This 'fictitious power' that is not a power at all in the normal sense of that word, is, in essence, a stand-in for the reactive component of current.

It is usually denoted by Q and its unit is *volt-ampere-reactive*, shortened as VAr . Thus, $Q = V_{rms} I_{rms} \sin\theta$ VAr where θ *is the phase angle by which the voltage phasor leads the current phasor. Therefore, the reactive power consumed by an inductive load* *is positive in sign and the reactive power consumed by a capacitive load is negative in sign by definition.*

Note carefully that the sign of *reactive component of current* and *reactive power* carried by that current are opposite.

Thus, an inductive load *draws* a 'negative reactive current' and *consumes* 'positive reactive power'. A capacitive load *draws* a 'positive reactive current' and *consumes* 'negative reactive power'. This is matter of convention and convenience rather than of necessity.

If a circuit element is *consuming* a certain amount of reactive power, it may equivalently be thought of as *delivering* negative of that amount of reactive power.

Thus an element that *draws* positive reactive power (*i.e.,* inductive *Q*) can be said to *deliver* negative reactive power (*i.e.,* capacitive *Q*).

Similarly, an element that *draws* negative reactive power (*i.e.,* capacitive *Q*) can be said to *deliver* positive reactive power (*i.e.,* inductive *Q*).

Thus a capacitor is a *source* of inductive reactive power and an inductor is a *source* of capacitive reactive power.

Notice that the Q value is the same as the amplitude of double-frequency power pulsation caused by reactive component of current.

One may easily show that (Apparent Power)^2 = $P^2 + Q^2$. Thus, a closed triangle can be constructed by treating apparent power, active power and magnitude of reactive power as its sides – the triangle will be called, obviously, the *power triangle*. This fact is also expressed in alternative forms as $(\text{VA})^2 = (\text{W})^2 + (\text{VAr})^2$ or $(\text{kVA})^2 = (\text{kW})^2 + (\text{kVAr})^2$.

It may also be noted that active power is alternatively called *real power* and *in-phase power*. Similarly, reactive power is also called *quadrature power*.

Many expressions are commonly employed to calculate reactive power. The first expression is used when the load circuit is a composite circuit containing many resistive and reactive elements. If $\boldsymbol{V} = V_{rms} \angle\phi_v$ and $\boldsymbol{I} = I_{rms} \angle\phi_i$ are the voltage phasor and current phasor at load terminals as per passive sign convention, then the Q delivered to the load circuit is $V_{rms}I_{rms} \sin(\phi_v - \phi_i)$ VAr. The other expressions are relevant when the voltage phasor and/or current phasor across a pure reactive element is known. In this case $(\phi_v - \phi_i)$ is assured to be 90^0 if the element is an inductor and -90^0 if the element is a capacitor. Then, the reactive power delivered to that element, *i.e., Q* is given by

$Q = V_{rms}I_{rms} = I_{rms}{}^2 X = \dfrac{V_{rms}{}^2}{X}$, where V_{rms} is the rms value of voltage across the element, I_{rms} is the rms value of current through the element and X is the reactance of that element. *Note that reactance of an inductor is a positive value and that of a capacitor is a negative value.*

8.10 Complex Power under Sinusoidal Steady-State Condition

Can we get these P and Q values from the voltage phasor and current phasor straightaway by multiplying them together? We will try. Let $\boldsymbol{V} = V_{rms}\angle\phi_v$ V rms be the voltage phasor and $\boldsymbol{I} = I_{rms}\angle\phi_i$ A rms be the current phasor; both specified as rms quantities. Then,

$$\boldsymbol{VI} = V_{rms}I_{rms}\angle(\phi_v + \phi_i) = V_{rms}I_{rms}\cos(\phi_v + \phi_i) + jV_{rms}I_{rms}\sin(\phi_v + \phi_i)\,.$$

We are not able to identify P or Q in the real and imaginary parts of this quantity since we know that $P = V_{rms}I_{rms}\cos(\phi_v - \phi_i)$ *and* $Q = V_{rms}I_{rms}\sin(\phi_v - \phi_i)$. But this observation prompts us to try $\boldsymbol{VI}^*$ instead of $\boldsymbol{VI}$.

$$\begin{aligned}\boldsymbol{VI}^* &= V_{rms}I_{rms}\angle(\phi_v - \phi_i) = V_{rms}I_{rms}\cos(\phi_v - \phi_i) + jV_{rms}I_{rms}\sin(\phi_v - \phi_i)\\ &= P + jQ\end{aligned}$$

Thus the quantity $\boldsymbol{VI}^*$ contains the active power as its real part and the reactive power as its imaginary part.

This quantity, $\boldsymbol{VI}^*$, is defined as *Complex Power* and is denoted by $\boldsymbol{S}$ with unit of *VA*.

Definition of Complex Power under sinusoidal steady-state condition

$$\boldsymbol{S}\,(\text{VA}) = \boldsymbol{VI}^* = P\,(\text{W}) + jQ\,(\text{VAr}) = \sqrt{P^2 + Q^2}\angle(\phi_v - \phi_i)\,\text{VA}$$

Therefore the magnitude of *complex power* is the apparent power and the angle of *complex power* is the angle by which the voltage phasor *leads* the current phasor. This angle is the same as the angle of driving-point impedance of the load circuit. $\boldsymbol{S}$, $P+j0$ and $0+jQ$ form a closed triangle in complex plane.

If the reader feels uncomfortable with the way the complex power was arrived at, let him be consoled by the fact that this was precisely how the expression for complex power was arrived at in the history of Electrical Engineering.

We had shown that both instantaneous power and average power are conserved in any circuit. Thus, active power under sinusoidal steady-state condition is a conserved quantity. That is, algebraic sum of active power delivered to all the elements of an isolated circuit under sinusoidal steady-state is zero. Does a similar conservation law hold for reactive power? It is possible to show that, it does, by using the expression for instantaneous power under sinusoidal steady-state conditions. Since both real part and imaginary parts of ***S*** are conserved, ***S*** itself is a conserved quantity. That is, algebraic sum of complex powers in all elements of an isolated circuit will be zero.

For an isolated circuit under single-frequency sinusoidal steady-state,

Power conservation under sinusoidal steady-state condition

$$\sum_{\substack{\text{over all}\\ \text{elements}}} \text{Active Power} = 0\ ; \quad \sum_{\substack{\text{over all}\\ \text{elements}}} \text{Reactive Power} = 0 \ \text{ and } \ \sum_{\substack{\text{over all}\\ \text{elements}}} \text{Complex Power} = 0\,.$$

Let $v(t) = V_m \cos(\omega t + \phi_v) = \sqrt{2}V_{rms} \cos(\omega t + \phi_v)$ and $i(t) = I_m \cos(\omega t + \phi_i) = \sqrt{2}I_{rms} \cos(\omega t + \phi_i)$ be the voltage and current as per passive sign convention in a load circuit. Let the series equivalent of the load circuit be R_S+jX_S and the parallel equivalent of the same circuit be R_P+jX_P. Then, the following table summarizes the various phasor quantities and their interrelations.

Table 8.10-1
Relationship Between Various Phasor Quantities in a Single-Phase System

Quantity	Relationship using peak values	Relationship using rms values	Unit
Voltage Phasor ***V***	$V_m\angle\phi_v$	$V_{rms}\angle\phi_v$	V
Current Phasor ***I***	$I_m\angle\phi_i$	$I_{rms}\angle\phi_i$	A
Complex Power ***S***	$0.5V_m I_m\cos(\phi_v-\phi_i)$ + $j0.5V_m I_m \sin(\phi_v-\phi_i)$	$V_{rms} I_{rms}\cos(\phi_v-\phi_i)$ + $jV_{rms} I_{rms} \sin(\phi_v-\phi_i)$	VA
Apparent Power $\lvert\boldsymbol{S}\rvert$	$0.5V_m I_m$	$V_{rms} I_{rms}$	VA
Active Power P	$0.5V_m I_m\cos(\phi_v-\phi_i)$	$V_{rms} I_{rms}\cos(\phi_v-\phi_i)$	W
Reactive Power Q	$0.5V_m I_m \sin(\phi_v-\phi_i)$	$V_{rms} I_{rms} \sin(\phi_v-\phi_i)$	VAr
Power Factor PF	$\cos(\phi_v-\phi_i)$	$\cos(\phi_v-\phi_i)$	-
Input Impedance ***Z***	$(V_m/I_m) \angle (\phi_v-\phi_i)$	$(V_{rms}/I_{rms}) \angle (\phi_v-\phi_i)$	Ω
R_S	$(V_m/I_m)\cos(\phi_v-\phi_i)$; or $2P/I_m^2$	$(V_{rms}/I_{rms})\cos(\phi_v-\phi_i)$ or P/I_{rms}^2	Ω
X_S	$(V_m/I_m)\sin(\phi_v-\phi_i)$ or $2Q/I_m^2$	$(V_{rms}/I_{rms})\sin(\phi_v-\phi_i)$ or Q/I_{rms}^2	Ω
R_P	$V_m/[I_m\cos(\phi_v-\phi_i)]$ or $2P/[I_m \cos(\phi_v-\phi_i)]^2$	$V_{rms}/[I_{rms}\cos(\phi_v-\phi_i)]$ or $P/[I_{rms} \cos(\phi_v-\phi_i)]^2$	Ω
X_P	$V_m/[I_m\sin(\phi_v-\phi_i)]$ or $2Q/[I_m \sin(\phi_v-\phi_i)]^2$	$V_{rms}/[I_{rms}\sin(\phi_v-\phi_i)]$ or $Q/[I_{rms} \sin(\phi_v-\phi_i)]^2$	Ω

Example : 8.10-1

Refer to Example : 8.6-4. Derive expressions for complex power delivered by the first source and complex power absorbed by the second source in a synchronous link and obtain approximate expressions for a situation when the phase difference between the sources is small and the difference in magnitude of voltages is small.

Solution

The synchronous link under consideration is shown in Fig. 8.10-1.

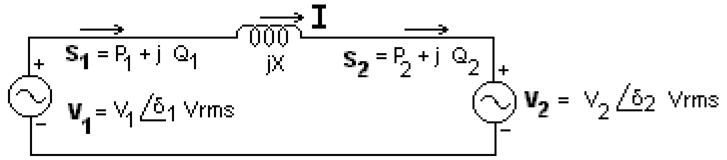


Fig. 8.10-1 A Synchronous Link

Let the current phasor from left to right be ***I***.

$$\boldsymbol{I} = \frac{V_1\angle\delta_1 - V_2\angle\delta_2}{jX} = \frac{V_1\angle(\delta_1 - \pi/2)}{X} - \frac{V_2\angle(\delta_2 - \pi/2)}{X}$$

$$\boldsymbol{S_1} = \boldsymbol{V_1}\boldsymbol{I}^* = V_1\angle\delta_1\left[\frac{V_1\angle-\delta_1 - V_2\angle-\delta_2}{-jX}\right] = \frac{V_1^2}{X}\angle\pi/2 - \frac{V_1V_2}{X}\angle(\pi/2 + \delta_1 - \delta_2)$$

$$= \frac{V_1V_2}{X}\sin(\delta_1 - \delta_2) + j\,\frac{V_1[V_1 - V_2\cos(\delta_1 - \delta_2)]}{X}$$

$$\therefore P_1 = \frac{V_1V_2}{X}\sin(\delta_1 - \delta_2)\ \text{watts}$$

$$Q_1 = \frac{V_1[V_1 - V_2\cos(\delta_1 - \delta_2)]}{X}\ \text{VArs}$$

Similarly,

$$\boldsymbol{S_2} = \boldsymbol{V_2}\boldsymbol{I}^* = V_2\angle\delta_2\left[\frac{V_1\angle-\delta_1 - V_2\angle-\delta_2}{-jX}\right] = \frac{-V_2^2}{X}\angle\pi/2 + \frac{V_1V_2}{X}\angle(\pi/2 + \delta_2 - \delta_1)$$

$$= \frac{V_1V_2}{X}\sin(\delta_1 - \delta_2) + j\,\frac{V_1[V_1\cos(\delta_1 - \delta_2) - V_2]}{X}$$

$$\therefore P_2 = \frac{V_1V_2}{X}\sin(\delta_1 - \delta_2)\ \text{watts}$$

$$Q_2 = \frac{V_2[V_1\cos(\delta_1 - \delta_2) - V_2]}{X}\ \text{VArs}$$

The link is purely inductive and we do not expect any loss of active power in the link. This is borne out by the fact that $P_1 = P_2$. The link inductor has voltage across it and current through it. Therefore this inductor will consume positive reactive power. Hence, we expect Q_2 to be less than Q_1. Let Q_L be the reactive power absorbed by the link inductor. Then,

$$Q_L = Q_1 - Q_2 = \frac{V_1^2 + V_2^2 - 2V_1V_2\cos(\delta_1 - \delta_2)}{X} = \frac{(V_1 - V_2)^2 + 2V_1V_2[1 - \cos(\delta_1 - \delta_2)]}{X}$$

Obviously, Q_L is a positive quantity and hence $Q_2 < Q_1$.

Special Case - $\delta = \delta_1 - \delta_2 << \pi/2$ and $V_1 = V + \Delta V$, $V_2 = V$

$$P_1 \approx \frac{V^2}{X}\delta\ \text{watts};\ Q_1 \approx \frac{V\times\Delta V + (\Delta V)^2}{X}\ \text{VArs}$$

$$P_2 \approx \frac{V^2}{X}\delta\ \text{watts};\ Q_2 \approx \frac{V\times\Delta V}{X}\ \text{VArs}$$

$$Q_L \approx \frac{(\Delta V)^2}{X}\ \text{VArs}$$

Thus, in a synchronous link operating with small phase difference and voltage magnitude difference between sources, the active power flows *from* the *leading source to* the *lagging source*. The amount active power will be proportional to phase difference (in radians) and will be relatively independent of voltage magnitude difference. Positive reactive power flow will take place *from* the *source with higher voltage magnitude to* the *source with lower voltage magnitude*. Reactive power flow in the link is proportional to voltage magnitude difference and is relatively insensitive to phase difference.

Example : 8.10-2

A 50 Hz, 63.5 kV rms substation provides power to a large industrial consumer through a 20 km long high voltage line that can be modeled by a resistance of 5 Ω and inductive reactance of 20 Ω. The voltage magnitude at receiving end is to be maintained at 63.5 kV. This is done by adjusting the sending end voltage magnitude by tap-changing transformers or otherwise. If the consumer draws 20 MW of power at 0.707 lag power factor, find (i) the magnitude of sending end voltage and power factor (ii) line current (iii) sending end active and reactive power (iv) active and reactive power absorbed by the line impedance (v) line power efficiency.

Solution

The magnitude of receiving end current = $20\times10^6 \div (0.707\times63.5\times10^3)$ = 445.5 A rms
The angle of current with respect to receiving end voltage phasor = $-\cos^{-1}0.707 = -45^0$
We take the receiving end voltage as the reference phasor.
Then, $\boldsymbol{V_R} = 63.5\angle 0^0$ kV rms and $\boldsymbol{I} = 445.5\angle{-45^0}$ A rms.

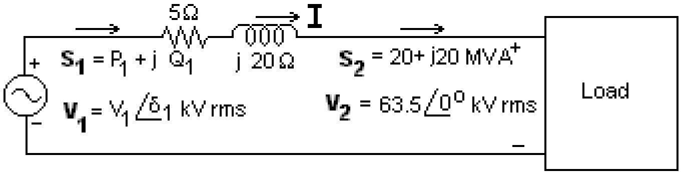


Fig. 8.10-2 Circuit for Example : 8.10-2

Active power delivered at receiving end = 20 MW
Reactive Power delivered at receiving end = 63.5 kV × 0.4455 kA × sin 45^0 = 20 MVAr
Active power consumed by the line impedance = $5\Omega \times (0.4455\text{ kA})^2$ = 1 MW
Reactive power consumed by the line impedance = $20\Omega \times (0.4455\text{ kA})^2$ = 4 MVAr
$\therefore$ Active power at sending end = 20+1=21 MW
$\therefore$ Reactive power at sending end = 20+4 = 24 MVAr
$\therefore$ Complex power at sending end, $\boldsymbol{S_1} = 21 + j\,24$ MVA
Since $\boldsymbol{S_1} = \boldsymbol{V_1}\,\boldsymbol{I}^*$, $\boldsymbol{V_1} = \boldsymbol{S_1} \div \boldsymbol{I}^* = (21 + j\,24)\times10^6 \div (445.5\angle{-45^0})^* = 71.6\angle 3.8^0$ kV rms.
The angle between sending end voltage phasor and current phasor = $3.8^0 - (-45^0) = 48.8^0$. Therefore sending end power factor = $\cos 46.74^0$ = 0.66 lag.

(i) Sending end voltage magnitude = 71.6 kV rms (12.75% above nominal value)
(ii) Sending end power factor = 0.66 lag
(iii) Sending end active power = 21 MW
(iv) Sending end reactive power = 24 MVAr
(v) Active power loss in line = 1 MW
(vi) Reactive power loss in line = 4 MVAr
(vii) Line power efficiency = 95.2%

Example : 8.10-3

If a capacitor is connected directly across the load at customer side in the problem stated in Example : 8.10-2 such that the receiving end current is at unity power factor with respect to receiving end voltage find the reactive power drawn by the capacitor and the capacitance value. Also calculate all the quantities calculated under Example : 8.10-2 and comment on the differences.

Solution

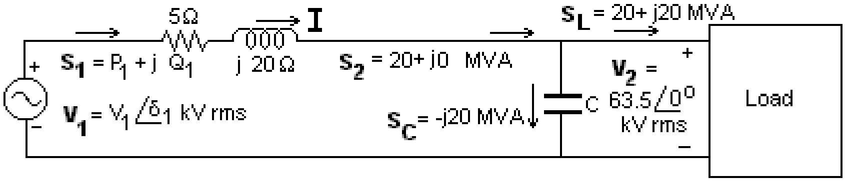


Fig. 8.10-3 Circuit for Example : 8.10-3

Refer to the circuit in Fig. 8.10-3. The capacitor should supply all the reactive power requirement of load, i.e., all of 20 MVAr if the current in the line at receiving end is to be at unity power factor. Therefore the current taken by capacitor will be $20\times10^6/63.5\times10^3$ = 315 A. Therefore capacitive reactance = $63.5\times10^3/315$ = 201.6 Ω. This value is $1/\omega C$ and $\omega = 2\pi\times50 = 100\pi$. Therefore C = 15.8 μF. With this capacitor in place, the current at receiving end of line will have a magnitude of $20\times10^6/63.5\times10^3$ = 315 A and its angle with respect to voltage will be 0^0. Then, $\boldsymbol{V_R} = 63.5\angle 0^0$ kV rms and $\boldsymbol{I} = 315\angle 0^0$ A rms.
Active power delivered at receiving end = 20 MW
Reactive Power delivered at receiving end = 63.5 kV × 0.315 kA × sin 0^0 = 20 MVAr

Active power consumed by the line impedance = $5\Omega \times (0.315\text{ kA})^2 = 0.5$ MW
Reactive power consumed by the line impedance = $20\Omega \times (0.315\text{ kA})^2 = 2$ MVAr
$\therefore$ Active power at sending end = 20+0.5=20.5 MW
$\therefore$ Reactive power at sending end = 0+2 = 2 MVAr
$\therefore$ Complex power at sending end, $\boldsymbol{S_1} = 20.5 + j\,2$ MVA
Since $\boldsymbol{S_1} = \boldsymbol{V_1}\boldsymbol{I}^*$, $\boldsymbol{V_1} = \boldsymbol{S_1} \div \boldsymbol{I}^* = (20.5 + j\,2)\times 10^6 \div (315\angle{-45^0})^* = 65.4\angle 5.6^0$ kV rms.
The angle between sending end voltage phasor and current phasor = $5.6^0 - (0^0) = 5.6^0$.
Therefore sending end power factor = $\cos 5.6^0 = 0.995$ lag.

(i)	Sending end voltage magnitude	= 65.4 kV rms (3% above nominal value)
(ii)	Sending end power factor	= 0.995 lag
(iii)	Sending end active power	= 20.5 MW
(iv)	Sending end reactive power	= 2 MVAr
(v)	Active power loss in line	= 0.5 MW
(vi)	Reactive power loss in line	= 2 MVAr
(vii)	Line power efficiency	= 97.6%

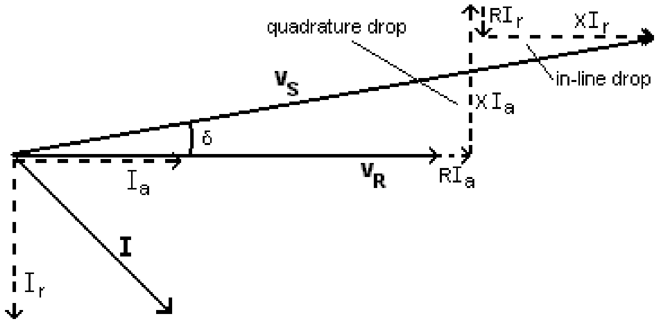


Fig. 8.10-4 Phasor Diagram of a line delivering power to a lagging load (Not to scale)

Comments on results arrived at in Example: 8.10-3

A load that draws power at a lag power factor results in a higher magnitude current in the line and in the source. Higher current magnitude results in higher voltage drop in the line, thereby requiring higher value of sending end voltage to maintain a specified receiving end voltage. Higher current magnitude results in higher active power loss in the line and thereby reduces power efficiency of the line.

The reactive component of load current undergoes a rotation by 90^0 to form the voltage drop in the line inductive reactance and results in an 'in-line' voltage drop. An 'in-line' voltage drop affects sending end voltage much more than a 'quadrature' voltage drop. See the phasor diagram in Fig. 8.10-4.

Therefore, reducing the reactive component of current drawn by a lagging load results in (i) lower current magnitude in the line and source (ii) lower line power loss and improved transmission efficiency (iii) lower voltage drop in the line. This reduction is effected by making a local capacitor act as a source of lagging reactive power required by the load. The line is thereby relieved from the task of supplying this reactive power.

This is called *capacitive compensation of lagging loads*. Capacitive compensation is routinely employed in Power Systems and Industrial Electrical Systems.

Example : 8.10-4

A 230 V rms source supplies two loads in parallel. The first one draws 10 kVA at 0.8 lag power factor. The second one draws 10 kW at 0.8 lead power factor. Find the source current rms value, complex power delivered by source and source power factor.

Solution

Complex power of first load = $10\times 0.8 + j\,10\times\sin(\cos^{-1}0.8) = 8+j6$ kVA
Complex power of second load = $10 - j\,(10/0.8)\times\sin(\cos^{-1}0.8) = 10-j7.5$ kVA.
Complex power is a conserved quantity.

Therefore complex power delivered by source = total complex power delivered to loads = 18-j1.5 kVA.

Source voltage is $230\angle 0^0$ V rms. Therefore, source current = $[(18-j1.5)\times 10^3/230]^*$ = $78.53\angle 4.76^0$ A rms.

Therefore, source current rms value is 78.53 A and source power factor is $\cos 4.76^0 = 0.997$ lead.

8.11 Sinusoidal Steady-State in Circuits with Coupled Coils

The physical basis for mutual inductance between two coils that share a common magnetic flux component has been dealt with in Chapter 1.

Flux linkage in a coil can be produced in two ways. The first method is to send a current through the coil. The flux linkage produced in the coil will be proportional to current through it and the proportionality constant is its self-inductance L. We now call this inductance as self-inductance because we need to distinguish it from mutual inductance. However, the qualifier 'self' is often dropped in practice and the word 'inductance' occurring alone is taken to mean 'self-inductance' by default.