Introduction

Electrical power generation, transmission and distribution employ sinusoidal voltage and current waveforms to carry power. That, in itself, is a sufficient reason for a detailed discussion on steady-state analysis of circuits containing *R*, *L*, *C* and *M*.

A sinusoidal waveform is completely specified by three parameters. For example, if $v_S(t) = A \sin(\omega t + \theta)$ volts, this waveform is completely specified by three numbers – A, ω and θ. Therefore, sending a pure sine wave from a transmitter to a receiver in a communications context is pointless because such a waveform can not carry any information other than that contained in just three numbers. And, for that matter, no waveform that is known completely beforehand can carry any information. Certain degree of uncertainty in the waveform to be transmitted is a precondition for information transmission from one location to another. Hence, the correct mathematical description of an information-bearing signal can only be a statistical description. But, despite this, the entire area of Electronic and Communication Engineering relies heavily on sinusoidal analysis of circuits and systems.

This raises two questions –

(i) Why did the Electrical Power Industry prefer sinusoidal waveform to any other waveform? (ii) Why does Electronics and Communications Engineering concern itself with sinusoidal analysis though a single-frequency sinusoid is hardly ever employed in a communication system?

These two questions are answered first in this chapter. Subsequently, the concepts of instantaneous power, average power, effective value of waveforms etc., are developed for sinusoidal waveforms as well as for other periodic waveforms.

7.1 Why Sinusoids?

An Electrical Power System, like any other electrical circuit, contains passive electrical elements like resistors, inductors, capacitors, mutually coupled coils and active elements in the form of independent voltage sources and current sources. The elements in a power system are modeled by linear time-invariant elements to a first degree of approximation. Resistors produce voltage drops across them that are proportional to the current flowing through them. Inductors demand a voltage that is proportional to rate of change of current through them. Capacitors demand a voltage across them that are proportional to the integral of current through them.

Electrical power system is usually voltage-driven. That is, independent voltage sources connected at various points in the system serve as sources of power in the system. The voltage sources connected at various source nodes (*i.e.,* the generating stations) drive currents through the series interconnections in the system, get modified by the voltage drops produced across various series path elements and appear as load voltage at load nodes in a modified form. Shunt elements connected from various nodes to the reference node in the system also influence this process of transformation of source voltages into load voltages.

Voltage drops across various elements thus modify the load voltage with respect to source voltage. These voltage drops are decided by a *scaling* of current by resistance value in the case of a resistor. It is decided by *derivative of current* in the case of an inductor and by *integral of current* in the case of a capacitor.

> Changes in waveshape can occur in linear time-invariant circuits containing dynamic elements.

A time-function retains its waveshape when multiplied by a constant. But, in general, it does not maintain its waveshape on *differentiation* and *integration*. Therefore, it follows that, in general, voltages and currents at various locations in an interconnected electrical network will have different waveshape even if all sources in the network have same waveshape.

That would surely complicate things in an Electrical Power System. In fact, it will not be a viable system at all. That prompts us to raise the question - is there any waveshape that will be *invariant* to time-domain differentiation and integration?

A generalised exponential function, $Ae^{\alpha t}$, has this property, as may be verified easily. The value of α can be complex. If α is real and positive, it represents a growing

Thus, sinusoidal signals have gained their pre-eminent position in Electrical Power Engineering since they belong to a special class of time-functions that preserve their waveshape on differentiation and integration.

A linear network excited by sinusoidal sources of a particular frequency will have sinusoidal voltages and currents at the same frequency everywhere in the system in the long run.

exponential and obviously is not suited in an electrical system that is expected to operate steadily for extended duration. If α is real and negative, it represents a real decaying waveform that tapers down to zero sometime or other. An electrical system excited by a set of such sources will settle down ultimately to a state in which all voltages and currents everywhere will be zero. Obviously, such source waveforms can not help the system to deliver power to loads in a steady manner for extended duration. If $\alpha = \gamma + j\omega$, the exponential function is a complex function of time and is given by $Ae^{\gamma t}\cos \omega t + j\, Ae^{\gamma t}\sin \omega t$. We can not generate an imaginary waveform in a physical system. But that problem can be solved by generating $0.5(Ae^{\alpha t} + Ae^{-\alpha t})$ which is equal to $Ae^{\gamma t}\cos(\omega t)$. Here too, the same objections we raised for a real exponential will hold if γ is non-zero. Therefore, we conclude that $0.5(Ae^{j\omega t} + Ae^{-j\omega t}) = A\cos \omega t$ or $-j0.5(Ae^{j\omega t} + Ae^{-j\omega t}) = A \sin \omega t$ are the possible choices for electrical power system source functions.

We observe that a *constant time-function* (*i.e.*, dc voltages and currents) is a special case of $Ae^{\alpha t}$ with $\alpha = 0$ and hence must satisfy the waveshape-preservation requirement. Thus, it should have been possible to generate, transmit and deliver steady power to loads in an electrical system by means of dc sources. It is indeed possible and that was how Electrical Power Industry started in late 19th century. But the problem associated with dc system was total inflexibility with respect to voltage and current levels used in the system. For instance, if the customer had to be given 220V at his premises, the generators had to generate 220V and all the interconnection system had to work at that voltage level. As the load level increases, the current flow everywhere become excessive and generation/transmission become inefficient due to resistive losses everywhere in the system. It would have been very convenient if generation could be done at a voltage level economical from the point of view of electrical machine design and operation. Similarly, it would have been convenient if the transmission of power through transmission lines could be done at high voltage level so that the current level and consequently losses in lines would decrease. But this calls for generation at low voltage level, transmission at high voltage level and consumption at low voltage level. An efficient 'voltage level conversion unit' is needed for this. Such voltage level conversion equipment for dc at high power levels was simply not available at the initial stages of evolution of power systems in late 19th century and early 20th century. And the same task turned out to be very easy in the case of a sinusoidal voltage system. A two-winding transformer can easily and efficiently change voltage levels in a system in the case of sinusoidal voltages. Thus, economic generation and transmission of huge quantities of electrical power became possible with sinusoidal voltage system and transformers. This is another reason why Electrical Power Industry is wedded to sinusoidal waveforms inalienably.

Why not DC instead of Sinusoids?

Advances in an area called Power Electronics resulted in power system loads becoming nonlinear in nature progressively from early 1980's. Power Electronic Equipment (ac to dc converters, thyristorised dc motor drives, variable speed ac drives, uninterruptible power sullplies, HVDC transmission systems etc.,) process the sinusoidal system power using nonlinear devices for a variety of purposes. Nonlinear loads draw *non-sinusoidal* currents from sinusoidal voltages. Non-sinusoidal current waveforms, subjected to differentiation and integration in various circuit elements, result in non-sinusoidal voltage drops across them. These non-sinusoidal voltage drops, in combination with the sinusoidal voltages produced at various generating stations, result in a non-sinusoidal voltage at load nodes in a power system. This is called the *Power System Harmonics Problem.* Another very important feature of sine waves comes to our help in analysing electrical systems that have non-sinusoidal periodic waveforms.

A broad class of periodic non-sinusoidal waveforms can be expressed as an infinite sum of sinusoidal waveforms with frequencies that are integer multiples of frequency of the periodic wave. For instance, a $\pm$ 1 volt square wave with 1 cycle per second frequency can be expressed as = $(4/\pi)[\sin 2\pi t + (1/3)\sin 6\pi t + (1/5)\sin 10\pi t + \dots\dots\dots + (1/n)\sin 2n\pi t + \dots\dots\dots]$. Infinite terms are needed; but the amplitude of sine wave decreases as the order of the term increases. It is possible to truncate this kind of series expansion of a periodic waveform to obtain reasonably accurate results in circuit analysis problems. Thus, series expansion of non-sinusoidal periodic waveforms in terms of sinusoidal waveforms and Superposition Theorem will help us to solve a circuit driven by such periodic waveforms provided we know how to solve it for a sinusoidal waveform. Thus even a power system under the influence of nonlinear loads can be analysed by sinusoidal analysis techniques with the help of this kind of series expansion for non-sinusoidal periodic waveforms.

Further, it turns out that, this series expansion of a periodic waveform leads to an expansion of even *aperiodic waveforms* in terms of sinusoidal waveforms as a limiting case. The series expansion of periodic waveform referred here is called *Fourier Series* and the expansion of an arbitrary aperiodic waveform in terms of sinusoidal functions is called *Fourier Transforms*. We deal with them extensively in later chapters of this book.

Fourier Series and Fourier Transforms along with Superposition Theorem help us to solve an electrical circuit driven by sources with arbitrary source functions by solving it for sinusoidal excitation. This is yet another factor that leads to pre-eminence of sinusoidal waveforms in circuit analysis. In fact, this is *the* reason why Electronics and Communication Engineers use sinusoidal analysis at all.

Sinusoidal waveforms and sinusoidal steady-state analysis of circuits is of great importance to electrical and electronic circuits due to the following reasons:

- Sinusoidal waveforms preserve their waveshape in linear circuits.
- Sinusoidal waveform render the voltage levels in electrical and electronic systems flexible so that optimization of sub-system performance in various parts of the system by choosing a suitable voltage level in that part of the system becomes possible. Transformers help us to realise this voltage flexibility.
- Periodic non-sinusoidal waveforms as well as a broad class of aperiodic waveforms can be expressed as a sum of sinusoids by Fourier series and Fourier transforms. Hence, circuit solution with such input waveforms can be obtained with relative ease if solution for a sinusoidal input is known.

7.2 The Sinusoidal Source Function

The sinusoidal voltage source function is generated in Synchronous Generators in Electrical Power Systems and by low power electronic oscillator circuits in Electronic Systems Communication Systems and Instrumentation/Measurement Systems. A source will have to be switched on at some point in time. Switching on a source may be a simple affair of switching on the dc power supply as in the case of an electronic sinusoidal oscillator circuit or switching on the ac mains to a function generator in the laboratory. It may be a complicated affair involving a sequence of steps as in the case of a bringing a generator in a Nuclear Power Station online.

Moreover, the sinusoidal voltage source may not start producing a sinusoidal output as soon as it is powered up. Usually it goes through a transient period during which its output builds up. During this period the output will not be a pure sine wave. After an initial period of adjustments it starts delivering sinusoidal output to whatever that is connected at its output.

Even if it produces a sine wave right from the instant at which it is powered up, it may not start at zero position in the waveform or at peak position in the waveform.

We assume in this section that the sinusoidal sources we are discussing here have been powered up in the past and have become steady. Moreover, we assume that they started at zero position on a sine wave when they were switched on. The concepts we evolve are not really dependent on these assumptions. The assumptions are made only to render clarity to the discussion that follow.

Amplitude, Period, Cyclic Frequency, Angular Frequency

Consider a single sinusoidal voltage source that was powered up in the past. We start observing the waveform of the voltage output in an oscilloscope from a particular point in time. We assign zero value to the time variable at the instant we start our observation of the waveform. Thus, the source was powered up in the past with respect to the instant at which we start observing it. The observed waveform is shown in Fig. 7.2-1. The time variable t is used in the horizontal axis.

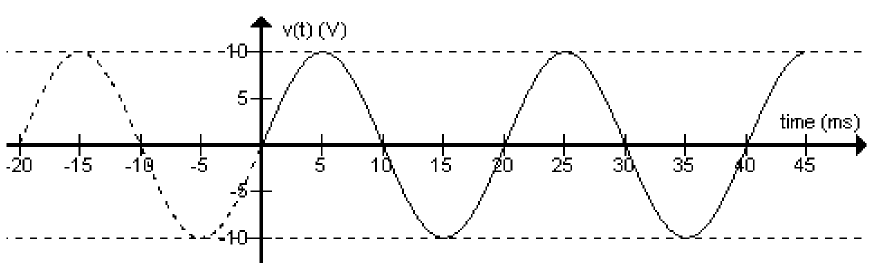

Fig. 7.2-1 Waveform of a Sinusoidal Voltage

The maximum positive value attained by the waveform is seen to be 10 volts and the maximum negative value attained also is 10 volts. This quantity is called the *amplitude* of the sinusoidal waveform. It so happened that the waveform was crossing zero from negative value to positive value at t = 0. This zero crossing is called the *positive-going zero crossing*. The zero crossing that happens when the voltage is crossing over from positive value to negative value is termed as *negative-going zero crossing*. That t = 0 happens to be a positive-going zero crossing is the result of a coincidence. But as a result of that coincidence, we are now free to write the voltage waveform that we observe from t = 0 onward as $v(t) = 10 \sin\omega t$. We need to work out the meaning and value of ω.

Period T of a sinusoidal waveform is the width of the basic repeating unit of the periodic waveform in the time-axis

We observe from Fig. 7.2-1 that the sinusoidal voltage completes one full cycle of variation in 20 ms. That is, if we start at any t and move through the waveform till we reach t+20 ms, we will find that the instantaneous voltage at t+20 ms is the same as the instantaneous voltage at t. Moreover, the shape of voltage variation in any ($t+n\times20$, $t+20+n\times20$) interval is same as in the interval (t, t+20) where unit of time is in ms and n is a positive integer. Thus, the waveshape is repetitive with its basic repeating unit decided by any 20ms interval. That is, the waveform is periodic from the instant we start observing it. The *period* of this waveform is 20ms. In general, period of a periodic waveform is the time interval needed to complete one full cycle of the waveform. The symbol, 'T' is used to represent the period of a periodic waveform. Or in other words, it is the width of the basic repeating unit of the periodic waveform in the time-axis.

Cyclic frequency f defined

The number cycles of variation that the waveform goes through in one second is defined as its *cyclic frequency*. The qualifier '*cyclic*' is often dropped when there is no cause for ambiguity or when the unit employed makes it clear that it is cyclic frequency that is being referred to. The unit of *cyclic frequency* is 'cycles-per-second' and is given a name Hertz. Hertz is written in short form as Hz. The shortened form 'cps' is also used to designate the unit of cyclic frequency. The cyclic frequency of the waveform in Fig. 7.2-1 is 1/20ms = 50 Hz. *Cyclic frequency* is usually indicated by the symbol 'f'.

Angular frequency ω defined as rate of change of angular argument of the sinusoidal waveform with respect to time

A sinusoidal function of an angle is periodic with a period of 2π radians. Thus the argument of the sinusoidal function in a sinusoidal waveform will go through an increment of 2π radians in one period. Therefore, the increment in the argument of the trigonometric function in one second will be $2\pi/T$ radians where T is the period of waveform (= $1/f$). This quantity, which represents the *rate of change of angle argument of the sinusoidal function with respect to time* is defined as the *angular frequency* or *radian frequency* of the sinusoidal waveform and is usually represented by the symbol ω (lower case omega). The unit of *angular frequency* is radians/sec, abbreviated as rad/s. Thus,

$$f = \frac{1}{T}\,\text{Hz},\quad \omega = \frac{2\pi}{T} = 2\pi f\ \text{rad/s}\ ,\ T = \frac{2\pi}{\omega} = \frac{1}{f}\,\text{sec}$$

The sinusoidal waveform of voltage source $v(t) = 10 \sin 100\pi t$ shown against t in Fig. 7.2-1 is redrawn against the angular argument ωt in Fig. 7.2-2 .

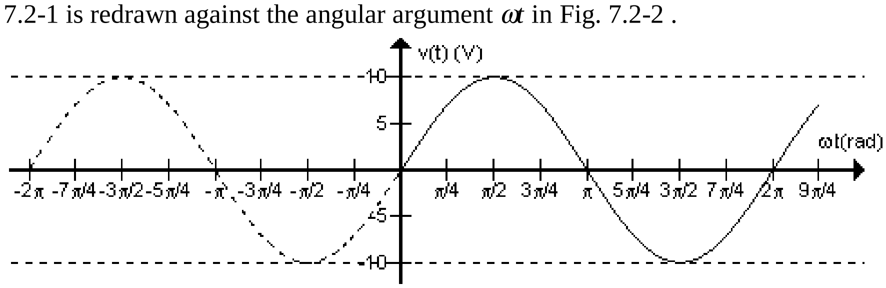

Fig. 7.2-2 Sinusoidal Waveform $v(t)$ Plotted Against ωt

A sinusoidal source voltage waveform that undergoes a positive-going zero crossing at t = 0 can be expressed as $v(t) = A \sin \omega t = A \sin (2\pi/T)\, t = A \sin 2\pi f t$ volts, where A is its *amplitude*, T is its period in seconds, f is its *cyclic frequency* in sec^{-1} (Hertz, Hz) and ω is its *radian frequency* or *angular frequency* in radians/sec (rad/s) unit.

Phase of a Sinusoidal Waveform

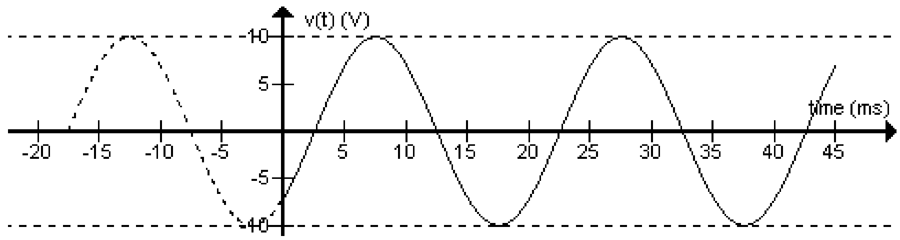

Fig. 7.2-3 Sinusoidal Waveform with a Non-zero Phase

Consider the observed sinusoidal waveform shown in Fig. 7.2-3. This waveform may be interpreted as a sinusoidal source powered up 17.5 ms prior to the start of observation. Assume that the source started delivering the sine wave output from zero value. The instantaneous value observed at the starting instant of observation is not zero, but –7.07 volts. The amplitude is observed to be 10 volts and the frequency is seen to be 50 Hz. Thus we can express this waveform as $v(t) = 10 \sin(100\pi t + \theta)$ where θ is an angle to be determined from the observed amplitude and instantaneous value at $t = 0$. The value at $t = 0$ is $10\sin\theta$ and this is seen to be –7.07 volts. Therefore $\theta = 45^0 = \pi/4$ rad. $\therefore v(t) = 10\sin(100\pi t - \frac{\pi}{4})$

This waveform is plotted against ωt in Fig. 7.2-4. Note that the instantaneous value of $v(t)$ at $\omega t = 100\pi t = \pi/4$ rad is zero as it should be.

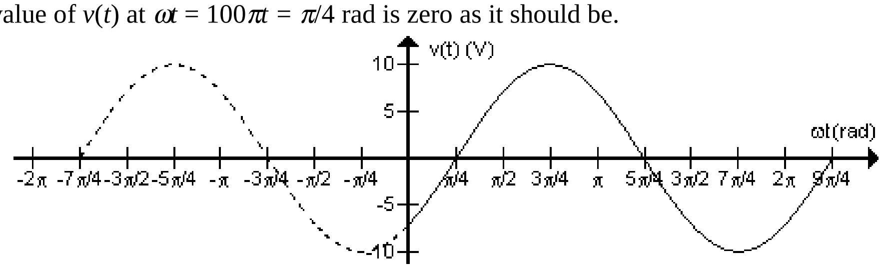

Fig. 7.2-4 A Sinusoidal Waveform with Non-Zero Phase Plotted Against ωt

The argument of trigonometric function in the expression for a sinusoidal waveform is always in radians. However, it is permitted to write the angle θ in degrees provided the symbol of degree is clearly shown. That is, $v(t)$ can be written as $10 \sin(100\pi t - 45^0)$ volts, but not as $10 \sin(100\pi t\text{-}45)$. In the second case, 45 will be interpreted with radian unit. Moreover, though the form $10 \sin(100\pi t - 45^0)$ volts is permitted, 45^0 has to be converted into radians before subtracting from the value of $100\pi t$ for some particular t before evaluating the sine function. In short, the form $10 \sin(100\pi t - 45^0)$ is allowed only as a *notation* and not for *function evaluation*.

The quantity θ in $v(t) = A \sin(\omega t + \theta)$ is defined as the *phase* of the sinusoidal function.

***Phase* has to be in radians when a trigonometric function is evaluated.**

Phase Difference Between Two Sinusoids

Consider the situation in Fig. 7.2-5. Two observers – A and B- use XY-Recorders A and B respectively to record the output from two sinusoidal voltage sources $v_1(t)$ and $v_2(t)$. A closes the two switches onto his Recorder at $t = 0$. t represents the time-axis chosen by him. B closes the switches onto his recorder at $t' = 0$. t' represents the time-axis chosen by B. $v_1(t)$ has an amplitude of V_{m1} and $v_2(t)$ has an amplitude of V_{m2}.

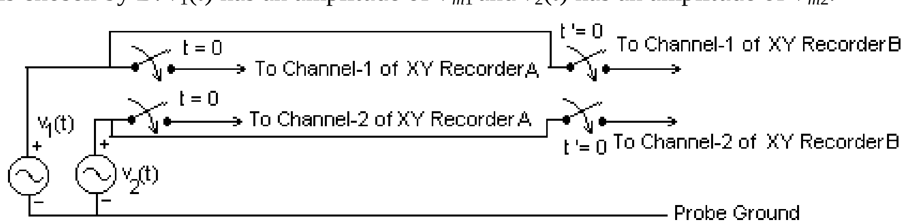

Fig. 7.2-5 Simultaneous observation of two sinusoidal sources by two observers with different starting Instants for observation

The waveforms recorded by Observer A are shown in (a) of Fig. 7.2-6 by the solid curve. The dotted curve shows the sinusoidal variation of sources prior to recording and will not show up in the recorder output. The waveforms in (a) are normalised with respect to their respective amplitude values to obtain the waveforms in (b).

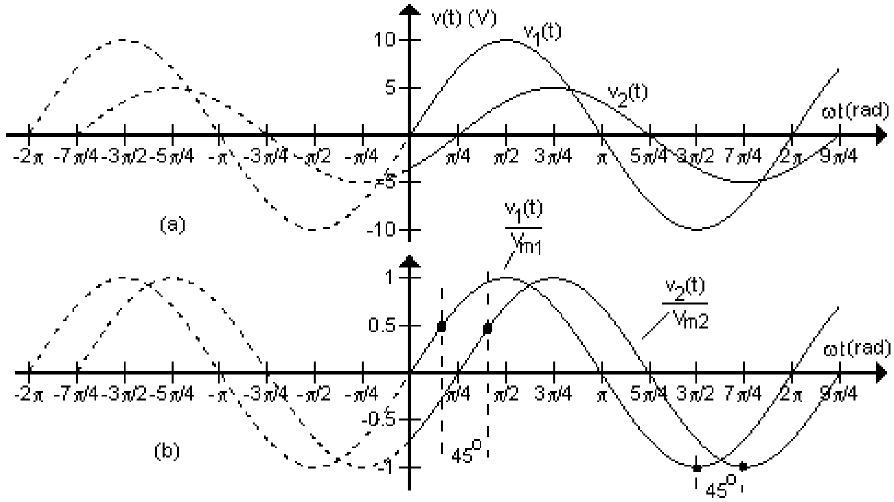

Fig. 7.2-6 Waveform Observation by Observer A

Two pairs of *similarly located waveform points within a cycle period* are located in the waveshape of $v_1(t)/V_{m1}$ and $v_2(t)/V_{m2}$ as shown in (b) of Fig. 7.2-6. Observer A notes that similarly located points in the two waveforms are separated by 45^0 in angle argument. A also notes that points on $v_2(t)$ come *after (in a visual sense)* similarly located points on $v_2(t)$.

The angular difference between similarly located points within a cycle period on two normalised sinusoidal waveforms (normalised with respect to their respective amplitude values) with same frequency is defined as the *phase difference* between them.

Similar observations recorded by B in $\omega t'$ axis is shown in Fig. 7.2-7.

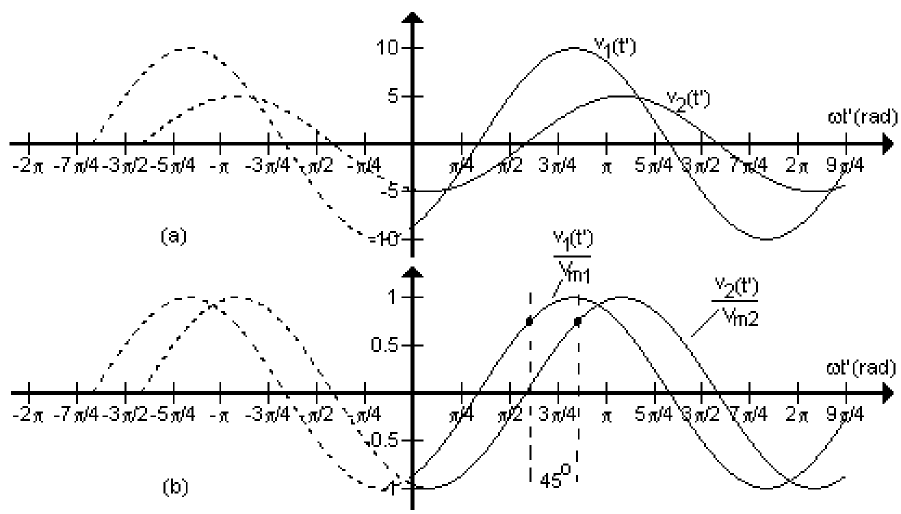

Fig. 7.2-7 Waveform Observations by Observer B

B too measures 45^0 angular separation between similarly located points on normalised $v_1(t)$ and $v_2(t)$. Moreover, B too observes that points on $v_2(t)$ come *after* similarly located points on $v_1(t)$.

The angular difference between similarly located points within a cycle period on two normalised sinusoidal waveforms (normalised with respect to their respective amplitude values) with same frequency is defined as the phase difference between them.The phase difference between two sinusoids is independent of choice of origin in t or ωt axis. The precedence relationship [i.e., which comes after (in a visual sense) which] between them in t or ωt axis too, is independent of choice of origin.

However, based on the observed amplitudes, values at origin and the position of first zero-crossing, A will write the sinusoidal functions as $v_1(t) = 10\sin 100\pi t$ volts and

$v_2(t)$ = 5 sin(100πt-45^0) volts. But, B will conclude that $v_1(t')$ = 10 sin(100$\pi t'$ – 60^0)volts and $v_2(t')$ =5 sin(100$\pi t'$ – 105^0)volts. Thus, the *phase* of $v_1(t)$ is 0^0 and *phase* of $v_2(t)$ is –45^0 as far as A is concerned. And they are –60^0 and –105^0 respectively as from B's point of view.

The phase of a sinusoidal waveform depends on the choice of origin in t or ωt axis. Phase difference between two sinusoidal waveforms at same frequency does not.

When a waveform point on a sinusoidal function $v_2(t)$ appears *after* a similarly located point on the waveform of another sinusoidal function $v_1(t)$ *with same frequency* , $v_2(t)$ is said to *lag* $v_1(t)$ in phase and the corresponding phase difference between them is called a *lag phase angle* under this condition.

Similarly, when a waveform point on a sinusoidal function $v_2(t)$ appears *before* a similarly located point on the waveform of another sinusoidal function $v_1(t)$ *with same frequency* , $v_2(t)$ is said to *lead* $v_1(t)$ in phase and the corresponding phase difference between them is called a *lead phase angle* under this condition.

It must be obvious that if $v_2(t)$ *lags* $v_1(t)$, then, $v_1(t)$ must necessarily *lead* $v_2(t)$. Moreover, if $v_2(t)$ *lags* $v_1(t)$, then, $v_2(t')$ will also *lag* $v_1(t')$ where t' is a new time variable as result of a different choice of origin.

Lag or Lead?

Similarly located points on two sinusoidal waveforms with same frequency have to be located within a period of the waveforms. But this leads to two choices for locating the point on the second waveform after having chosen a point on the first waveform. Refer Fig. 7.2-8.

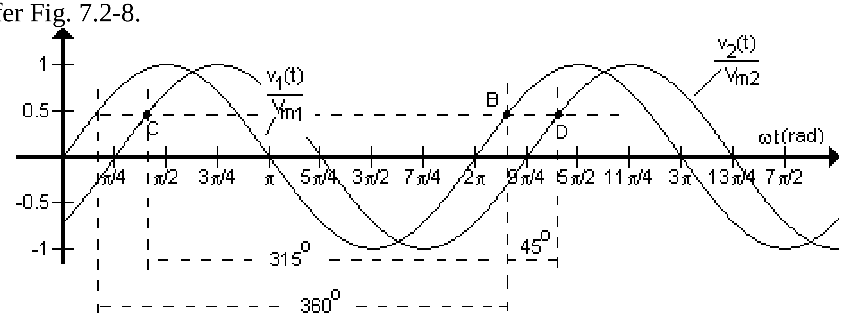

Fig. 7.2-8 Relationship between Phase Lag and Phase Lead

We locate the point B on normalised $v_1(t)$ first. We are free to locate the similarly located point on normalised $v_2(t)$ on either side of B within a span of 2π radians or 360^0. This gives us two choices – point C and point D on the second waveform. If we choose point C, we can conclude that $v_2(t)$ *leads* $v_1(t)$ by 315^0. If we choose point D, we conclude that $v_2(t)$ *lags* $v_1(t)$ by 45^0. Therefore a lag angle of θ radians and a lead angle of (2π-θ) radians mean the same. As a convention, we favor the angle that turns out to be less than 180^0 (or π radians). Thus, in Fig. 7.2-8, we will term it as a lag angle of 45^0.

Phase Lag/Lead versus Time Delay/Advance

But doesn't it leave a certain ambiguity about which waveform is *after* the other? It does. If we accept the point C on $v_2(t)$ in Fig. 7.2-8 as the point corresponding to point B on $v_1(t)$, we conclude that $v_2(t)$ comes *before* $v_1(t)$. Similarly, If we accept the point D on $v_2(t)$ as the point corresponding to point B on $v_1(t)$, we conclude that $v_2(t)$ comes *after* $v_1(t)$. However, note carefully that we had been careful to keep the precedence relationships –*before* and *after* – only in relation to our visual perception of the waveform plots. We have not yet ascribed temporal significance to these terms. That is, we have *not* stated till now that if a waveform $v_2(t)$ comes *after* $v_1(t)$ *in a visual sense*, then, $v_2(t)$ started later than $v_1(t)$ in time. Or, in other words, we have not correlated the *phase difference* between two waveforms with *time delay* or *time advance* between them. The term '*phase lag*' tends to give us an impression that the waveform that *lags* behind suffered some *time delay* with respect to the other waveform. But this impression can be wrong. Similarly, the waveform that *leads ahead* of another waveform did not necessarily start earlier. The reader is cautioned against equating a '*phase lag*' with a

Phase Lag/Lead versus Time Delay/Advance

Phase lag/lead between various voltages and currents in a Power System has profound implications in the economic operation of the system.

Phase delay and time delay between various sinusoidal waveforms in an electronic system or communication system has great significance in terms of waveform distortion and loss of information contained in a waveform. Therefore, Electrical Power Engineers pay a great deal of attention to *phase lag/lead* between waveforms whereas Electronics and Communication Engineers place even higher emphasis on *phase delay/advance* and *time delay/advance* between waveforms.

Hence these terms are discussed in detail in this sub-section.

'*time delay*' and a '*phase lead*' with a '*time advance*' indiscriminately. There are situations in which a '*phase lag (lead)*' implies a '*time delay (advance)*' – in that case, we will term the phase lag (lead) as '*phase delay (advance)*'. And there are situations in which *lag/lead* can not be uniquely correlated to *delay/advance* in time-domain.

Relationship between phase lag, phase delay and time delay

Consider the two waveforms $v_1(t)$ and $v_2(t)$ in (a) of Fig. 7.2-9. Additional information in the form of dotted curves is also shown in (a). The frequency of waveform is 50 Hz. The waveforms in (a) show that the source $v_1(t)$ started generating a sinusoidal voltage at 20 ms before the observation started and $v_2(t)$ started only 2.5 ms later. It is also clear that $v_2(t)$ lags $v_1(t)$ by 45^0. Thus, a *time delay* of 2.5 ms has resulted in a *phase lag* of 45^0. Since we know from the additional information provided in the form of dotted curves that the phase difference between two sources resulted from a time delay, we can term this 45^0 phase lag as a 45^0 *phase delay* too. Obviously the following relation between *time delay* and *phase delay* holds.

$$\text{phase delay}\,\phi \text{ in radians} = \omega \times \text{time delay}\, t_d \text{ in seconds}$$
$$i.e.,\ \phi = \omega t_d$$

Similar statement can be arrived at in the case of *time advance* too – provided we know from information other than we obtained from observing the two waveforms from t = 0 that the observed phase difference is due to a *time advance*.

$$\text{phase advance}\,\phi \text{ in radians} = \omega \times \text{time advance}\, t_d \text{ in seconds}$$
$$i.e.,\ \phi = \omega t_d$$

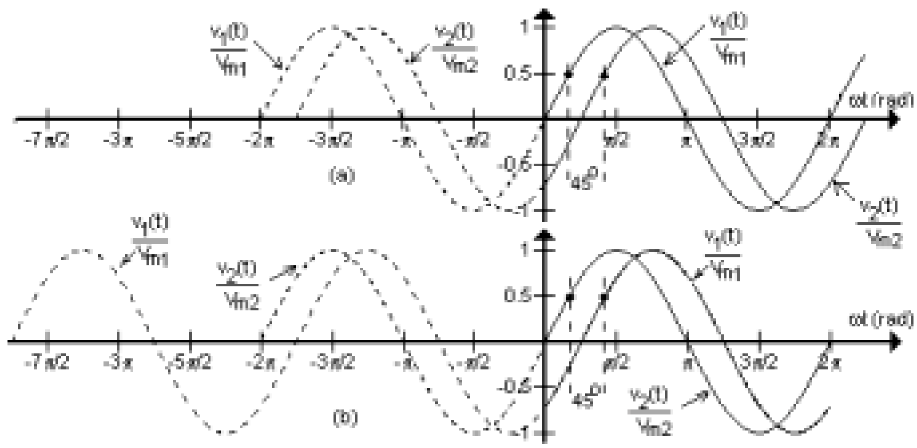

Fig. 7.2-9 Illustrating Phase Delay and Time Delay

Additional information is required to translate an observed *phase lag/lead* relationship between two sinusoidal waveforms into a *phase delay/advance* (equivalently, *time delay/advance*) relationship between them.

'Phase lag' is not necessarily a 'phase delay' and 'phase lead' is not necessarily a 'phase advance'.

'Phase lag' does not necessarily imply 'time delay' and 'phase lead' does not necessarily imply 'time advance'.

Now, consider the waveforms in (b) of Fig. 7.2-9. Here $v_1(t)$ started 17.5 ms earlier than $v_2(t)$. But if we go only by the observation from t = 0 onwards, we will conclude that $v_2(t)$ *leads* $v_1(t)$ by 45^0 or equivalently $v_2(t)$ *lags* $v_1(t)$ by 315^0, and, as per the agreed convention, we will settle for '$v_2(t)$ *leads* $v_1(t)$ by 45^0'. But, in the light of the additional information given in the form of dotted curves, a translation to the effect that $v_2(t)$ started 2.5 ms earlier than $v_1(t)$ will be in error. Actually $v_2(t)$ started 17.5 ms (315^0) after $v_1(t)$ and hence *phase delay* of $v_2(t)$ is 315^0 and *time delay* of $v_2(t)$ is 17.5 ms with respect to $v_1(t)$. The conclusion from observation from t = 0 onwards can also be stated as $v_1(t)$ *lags* $v_2(t)$ by 45^0. Again, a translation to the effect that $v_1(t)$ started 2.5 ms after $v_2(t)$ started is wrong in the light of additional information given. Actually $v_1(t)$ started 17.5 ms before $v_2(t)$ started and hence $v_1(t)$ has a *phase advance* of 315^0 and a *time advance* of 17.5 ms with respect to $v_2(t)$.

The additional information needed to decide *time delay/advance* from *phase lag/lead* is not usually available in the case of multiple sinusoidal source waveforms in a complex electrical system. But then, we do not need the *time delay/advance* information in Electrical Power Systems usually.

There is one situation in which this additional information needed is invariably available. Consider a situation in which a sinusoidal voltage source is applied to a linear electrical network at t = 0. The circuit variables respond to this excitation and assume pure sinusoidal variation at the same frequency as that of sinusoidal excitation in the long run. There will be a definite phase difference between a response variable (may be a current in some element or a voltage across some element) and the source function. *No physical system can produce a response before the excitation is applied to it.* Response always *follows* the excitation in a *physical system* and can not precede excitation. This intuitively obvious fact is known as the '*law of causality*' for physical systems. Thus, law of causality of physical systems effectively states that the response will be *delayed* with respect to excitation.

Example : 7.2-1

Two sinusoidal waveforms, $x(t)$ and $y(t)$, recorded from $t = 0$ are shown in Fig. 7.2-10. $x(0) = -2.571$ and $y(0) = 1.25$ (i) Express $x(t)$ and $y(t)$ as sine functions and identify their amplitude, period, cyclic frequency, radian frequency, phase and phase difference. (ii) If no additional information is available, list all possible time delay/advance relations between the two assuming that both started at positive-going zero-crossing position. (iii) If $x(t)$ is voltage waveform that was powered up and applied to a linear electrical circuit long back in the past and $y(t)$ is the current flow in some element in that circuit, find the time delay/advance between the two and phase delay/advance between the two.

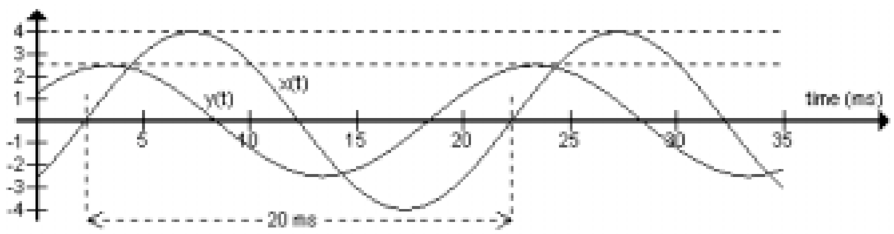

Fig. 7.2-10 Sinusoidal Waveforms for Example : 7.2-1

> **Phase lag and time delay in Circuits**
>
> The response sinusoid in an electrical circuit will always be delayed with respect to the excitation sinusoid quite regardless of whether the phase difference is a lag angle or lead angle. A phase lead that the response variable exhibits with respect to excitation variable in a physical electrical network has to be understood as a phase delay that is more than π radians and a time delay that is more than half-period.
>
> The reader is cautioned against the commonly made mistake of assuming that the apparent phase lead exhibited by a response variable implies a phase advance or time advance.

Solution

(i) The period of both waveforms, $T = 20$ ms. Therefore $f = 50$ Hz and $\omega = 100\pi$ rad/s. Amplitude of $x(t)$ is 4 and amplitude of $y(t)$ is 2.5.

Therefore, $x(t) = 4\sin(100\pi t + \theta_x)$ and $y(t) = 2.5\sin(100\pi t + \theta_y)$ where θ_x and θ_y are the phases of $x(t)$ and $y(t)$ respectively. The values of $x(t)$ and $y(t)$ at $t = 0$ are given as –2.571 and 1.25 respectively.

$\therefore \sin\theta_x = -2.571/4 = -0.6428 \Rightarrow \theta_x = -40^0$ or -140^0.

The choice between -40^0 and -140^0 is made by observing that if it is -40^0, the first zero-crossing of the waveform that takes place at $\omega t = -\theta_x$ will take place within the first quarter cycle and if it is -140^0 the first zero-crossing after $t = 0$ will take place after first quarter cycle. In the present case $x(t)$ crosses zero within first 5 ms (first quarter cycle) and hence $\theta_x = -40^0$.

$\therefore x(t) = 4\sin(100\pi t - 40^0)$

Similarly, $\sin\theta_y = 1.25/2.5 = 0.5 \Rightarrow \theta_y = 30^0$ or 150^0. Now the first zero crossing will take place at $\omega t = 150^0$ position (*i.e.*, in the second quarter cycle) if $\theta_y = 30^0$ and it will take place at $\omega t = 30^0$ position (*i.e.*, in the first quarter cycle) if $\theta_y = 150^0$. Since it takes place within the second quarter cycle in the present case, $\theta_y = 30^0$.

$\therefore y(t) = 2.5\sin(100\pi t + 30^0)$

The phase difference between the two waveforms has a magnitude of $30^0-(-40^0) = 70^0$ and $x(t)$ *lags* $y(t)$ by 70^0. Equivalently, $y(t)$ *leads* $x(t)$ by 70^0. Or equivalently, $x(t)$ *leads* $y(t)$ by 290^0 and $y(t)$ *lags* $x(t)$ by 290^0.

(ii) 70^0 phase difference translates to $20\times70/360 \approx 3.89$ ms time interval. The possibilities are:

(a) $y(t)$ started $(3.89+ 20n)$ ms *before* $x(t)$ and therefore $y(t)$ has a *phase advance of* $(70^0+ n360^0)$ with respect to $x(t)$ where $n = 0, 1, 2, 3,$......... . This may also be restated as $x(t)$ has a *phase delay of* $(70^0+ n360^0)$ with respect to $y(t)$.

(b) $x(t)$ started $(16.11+ 20n)$ ms *before* $y(t)$ and therefore $x(t)$ has a *phase advance of* $(290^0+ n360^0)$ with respect to $y(t)$ where $n = 1, 2, 3,$......... . This may be restated as $y(t)$ has a *phase delay of* $(290^0+ n360^0)$ with respect to $x(t)$.

(iii) Now, $x(t)$ is the *cause* and $y(t)$ is the *effect* in a physical electrical circuit. Therefore, by law of causality, $y(t)$ can have only a *phase delay* with respect to $x(t)$. Therefore, $y(t)$ has a *phase delay of* $(290^0+ n360^0)$ with respect to $x(t)$.

However, this does not mean that the circuit waited for (16.11+ 20n) ms after the voltage waveform was applied to it doing nothing in that interval and then started producing a sinusoidal current in the element under consideration! What happens in the electrical circuit is that, as soon as the voltage waveform applied, the circuit starts a mixed response that includes even non-sinusoidal terms caused by the electrical inertia of the circuit. This period is called the transient period. The non-sinusoidal components in the response die down to zero as time progresses. After a sufficiently long duration decided by circuit parameters, a steady-state comes up in the circuit in which all the response variables become pure sinusoidal waveforms at the same frequency as that of excitation. And, by that time the current $y(t)$ would have acquired a steady *phase delay* of $(290^0+ n360^0)$ with respect to the applied voltage $x(t)$. The value of n that is applicable can be obtained only if the circuit is known in detail and a full circuit solution is obtained.

7.3 Instantaneous Power in Periodic Waveforms

A waveform $v(t)$ is said to be periodic with a periodicity of T seconds if $v(t+nT) = v(t)$ for all integer values of n and for all t. This implies that it must be possible to identify a basic section of the waveform that lasts for T seconds and that repeats to infinite extent into the past and into the future. Thus, a waveform is strictly periodic only if it is ever existent. But, in practice waveforms are switched on at some definite time instant. Such switched waveforms can not be called periodic waveforms in the strict sense of definition of periodicity. However, we can view them as periodic waveforms for circuit analysis purposes provided we focus our attention to time instants located far away from the instant at which the waveform was switched on.

Consider a two-terminal electrical element with the current and voltage variables marked as per *passive sign convention* in Fig. 7.3-1.

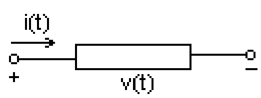

Fig. 7.3-1 A Two-terminal Element with Voltage and Current Marked as per Passive Sign Convention

The voltage difference v_{AB} between two points A and B is the work to be done in moving +1 coul of charge from B to A. Energy has to be spent in carrying charge from a lower potential point to higher potential point. Similarly, energy is released when a charge is allowed to fall through a higher potential point to lower potential point. The amount of charge that went through the element from a higher potential point to lower potential point in one second is given by $i(t)$. Therefore the product of $v(t)$ and $i(t)$ must be the energy released into element in one second. The rate of change of energy is defined as *instantaneous power* and denoted by $p(t)$.

Instantaneous power delivered to an element is defined as $p(t) = v(t)\, i(t)$

Therefore, instantaneous power delivered to a two-terminal element, $p(t) = v(t)\, i(t)$, where $v(t)$ and $i(t)$ are the element variables defined as per passive sign convention.

Then, energy *delivered to* a two-terminal element is obviously given by

total energy dissipated in the element from infinite past to t

$E(t)=\int_{-\infty}^{t} p(t)\,dt=\int_{-\infty}^{t} v(t)i(t)\,dt = E(0)+\int_{0}^{t} v(t)i(t)\,dt$ where $E(0)$ is the total energy dissipated in the element from infinite past to $t = 0$.

And the relation between the energy function $E(t)$ and the instantaneous power $p(t)$ is given by $p(t)=\dfrac{dE(t)}{dt}$.

Let $E_i(t)$ be the energy dissipation function (*i.e.*, the net energy delivered from $-\infty$ to t) of the i^{th} element in a b element electrical circuit. Then the total energy dissipation function of the circuit is $E_T(t) = \sum_{i=1}^{b} E_i(t)$. The circuit considered as a whole is an isolated system and the *total energy in an isolated system is a constant* by Conservation of Energy.