Introduction

We, at this point in our study of circuit analysis, know how to analyse a linear time-invariant circuit with initial conditions to obtain its transient and steady-state performance by employing time-domain techniques or frequency-domain techniques. Further, we know how to do it efficiently by judicious application of some powerful circuit theorems.

Of course, Circuit Theory does not end there. In fact, it is only the beginning. However, a close study of the previous 15 chapters of the text would have imparted to the reader the capability to analyse any linear time-invariant circuit of reasonable complexity.

There is nothing special about a two-port network from this point of view. It is just another linear time-invariant circuit that can be analysed by the techniques evolved in the first 15 chapters of this text.

However, evolving special analysis procedures for two-port networks (we will see soon what they are) makes the circuit analysis procedure more efficient and more elegant.

There are practical application circuits in which a major portion of the circuit does not undergo any change in structure or in parameters and the remaining portion undergoes changes in structure and/or parameters. The circuit is to be analysed for various parameter sets. A case in point would be an electronic amplifier. Maybe we have designed an amplifier and we would like to analyse its performance (voltage gain, input impedance, bandwidth, efficiency etc.) for different loading conditions at the output and for different driving conditions at the input. The structure, interconnections and parameters inside the amplifier do not change. But what is connected at the input and at the output of the amplifier change from one circuit analysis problem to another. In this case, should we really set up all the circuit equations for each configuration and solve the entire set of equations repeatedly for each input drive – output load configuration? Can't we abstract the circuit behaviour of the unchanging portion of the circuit (*i.e.,* the amplifier itself) in terms of a condensed set of equations and use this condensed set of equations to solve the entire circuit repeatedly with different input and output conditions? If we could, that would make the analysis procedure efficient.

In fact, we can do precisely that, provided the unchanging portion of the circuit satisfies certain conditions. The amplifier in this discussion has a pair of terminals at which an input source can be applied. It has another terminal pair across which a load or a two-terminal load network can be connected. These are the two terminal-pairs at which the remaining portion of the integral circuit is allowed to interact with the amplifier. *We generalise these ideas to define a linear time-invariant two-port network as a linear network that contains no independent sources and has two pairs of terminals at which it can interact with the external world with no interaction permitted with the external world except through these two terminal pairs.*

A "linear time-invariant two-port network" contains linear, lumped circuit elements and *does not contain independent sources*.

It has two terminal pairs identified for interaction with other circuits external to it. These two terminal pairs may be accessed in order to apply excitation source to the network or to measure response of the network.

Each such terminal-pair at which an excitation may be applied or a response may be measured is called a "port".

The port at which the input source is customarily applied is called the input port and the port at which the output is usually measured is termed as the output port. Usually the left side port is taken to be the input port and right side port is taken to be the output port while drawing the circuit diagrams of two-port networks unless specified otherwise.

The input port and output port can have one terminal common between them. In that case it will be a three-terminal two-port network. Otherwise, it will be a four-terminal two-port network.

Note carefully that the only interaction that the definition of two-port network permits is the interaction through the ports. This implies that the outside world can not interact with the network except through these two ports. For instance, the external circuitry should not have magnetic or electric coupling with any element inside the two-port network. Similarly, the external circuitry should not have any coupling with the two-port network through dependent sources. In fact, no kind of coupling – optical, electro-acoustic, piezo-electric etc., - are permitted between the external circuit and the two-port network. If such coupling exists, it is not possible to partition the circuit into a two-port network interacting with an external network.

A two-port network can interact with the external world *only through the ports* !

The external variables, *i.e.*, the port-voltages and port-currents of the two-port network satisfying the above conditions, can be described by a condensed set of equations. In fact, there are only four variables – two voltages and two currents. And, there will only be two equations tying up these four variables. These two equations will describe the behaviour of the two-port network as far as its interaction with the external circuit is concerned. Detailed modeling of the internal details of the two-port network is needed just once – and that is for finding out the two describing equations for the two-port network. Once these two equations are obtained, the two-port network may subsequently be replaced by this equation set for repeated analysis with varying external circuit parameters.

This chapter deals with two-port network equations and their applications to passive two-port electric filters.

16.1 Describing Equations and Parameter Sets for Two-Port Networks

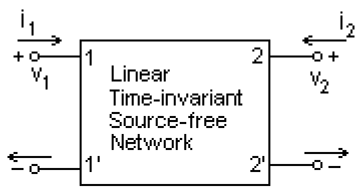

Fig. 16.1-1 A linear time-invariant two-port network

A linear time-invariant two-port network with all the *port variables* identified is shown in Fig. 16.1-1.

The port-currents are shown to enter the network at both ports. This is a matter of tradition and we have no reason to depart from this tradition. The two port-voltage variables and two port-current variables are instantaneous variables and are functions of time in general. However, they could very well be phasors or Laplace transforms if the two-port network is drawn as phasor equivalent circuit or *s*-domain equivalent circuit respectively.

The current entering a port through a terminal will leave through the other terminal (Why?)

Now we consider a general network that can be viewed as a cascade of three sub-networks. The first network, N_A , drives a linear time-invariant two-port network N_B and the third network N_C loads the two-port network at its second port. Both N_A and N_C may contain independent sources and for simplicity we take them to be linear. We assume that (i) this cascaded network has a unique solution and that (ii) N_A, N_B and N_C interact only through the terminals of interconnection. This cascade is shown in Fig. 16.1-2.

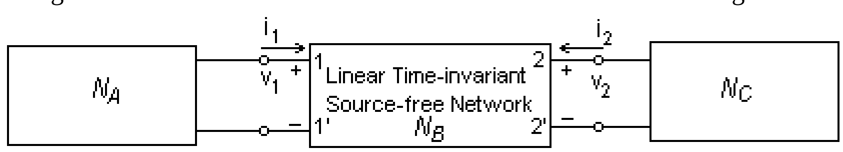

Fig. 16.1-2 A linear two-port embedded in a cascaded network

Short-Circuit Admittance Parameters for a Two-Port Network

This network cascade in Fig. 16.1-2 satisfies the conditions for applying Substitution Theorem. We apply this theorem to replace the network N_A by an independent voltage source to arrive at the circuit shown in Fig. 16.1-3. We assume that the circuit in Fig. 16.1-3 has a unique solution. Linear circuits generally have unique solution except in certain rare instances of degeneracy. We forget about such rare degenerate circuits. (After all, this is an introductory text on Circuits!).

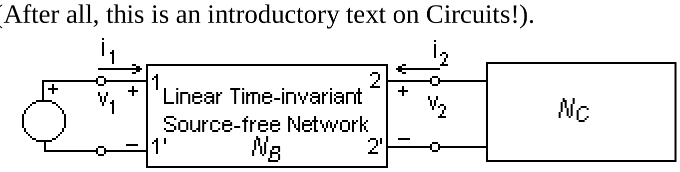

Fig. 16.1-3 Circuit in Fig. 16.1-2 after applying Substitution Theorem

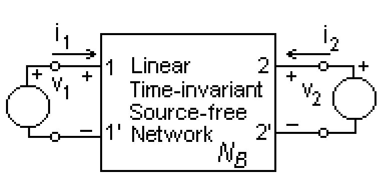

Fig. 16.1-4 Circuit in Fig. 16.1-2 reduced to a linear two-port driven by two voltage sources

But, if this circuit has a unique solution, we can apply Substitution Theorem once more to replace the network N_C by an independent voltage source v_2 without affecting the circuit solution anywhere in network N_B as shown in Fig. 16.1-4.

Therefore, we can solve for i_1 and i_2 in the network cascade in Fig. 16.1-2 by developing equations for the voltage-driven two-port network in Fig. 16.1-4.

Any response variable in a linear time-invariant circuit can be expressed as a linear combination of excitation functions. This is a consequence of linearity of the circuit and we have been using this very important principle right from Chapter 4 of this text. The excitation functions are v_1 and v_2 in this case. The response variables are i_1 and i_2. Therefore, we can express i_1 and i_2 as

$$\begin{aligned} i_1 &= y_{11}v_1 + y_{12}v_2 \\ i_2 &= y_{21}v_1 + y_{22}v_2 \end{aligned} \quad (16.1\text{-}1)$$

Defining equation for *y-parameters*

where y_{11}, y_{12}, y_{21} and y_{22} are real-valued proportionality constants in the case of memoryless circuits, functions of $j\omega$ in the case of phasor equivalent circuits and functions of s in the case of s-domain equivalent circuits. They connect voltage variables to current variables and hence have dimension of admittance. This was anticipated in using '*y*' to designate them. They are called *y-parameters* of the two-port network and the network description in Eqn. 16.1-1 is called *y-parameter description* for the two-port.

Obviously, the *y-parameters* of a two-port can be evaluated by analysing the two-port in isolation. Once the parameters are obtained, the same set of parameters can be employed to describe the two-port in terms of Eqn. 16.1-1 to analyse the network cascade in Fig. 16.1-2 for a variety of networks N_A and N_C.

How do we obtain these parameters? We interpret the Eqn. 16.1-1 in the following manner to obtain them.

$$\begin{aligned} i_1 &= y_{11}v_1 + y_{12}v_2 \\ i_2 &= y_{21}v_1 + y_{22}v_2 \end{aligned} \quad (16.1\text{-}2)$$
$$\therefore y_{11} = \left.\frac{i_1}{v_1}\right|_{v_2=0},\ y_{12} = \left.\frac{i_1}{v_2}\right|_{v_1=0},\ y_{21} = \left.\frac{i_2}{v_1}\right|_{v_2=0},\ y_{22} = \left.\frac{i_2}{v_2}\right|_{v_1=0}$$

y_{11} is the input admittance at port-1 with port-2 kept shorted.

y_{12} is the reverse transfer admittance from port-2 to port-1 with port-1 kept shorted.

y_{22} is the input admittance at port-2 with port-1 kept shorted.

y_{21} is the forward transfer admittance from port-1 to port-2 with port-2 kept shorted.

Hence, we note that determination of the *y-parameters* involve shorting one or the other port. This is why they are also called *Short-Circuit Admittance Parameters* of the two-port.

The defining equations in Eqn. 16.1-2 assumed a memoryless linear time-invariant two-port network. The equations get modified as

$$y_{11}(j\omega) = \left.\frac{\boldsymbol{I_1}}{\boldsymbol{V_1}}\right|_{\boldsymbol{V_2=0}},\ y_{12}(j\omega) = \left.\frac{\boldsymbol{I_1}}{\boldsymbol{V_2}}\right|_{\boldsymbol{V_1=0}},\ y_{21}(j\omega) = \left.\frac{\boldsymbol{I_2}}{\boldsymbol{V_1}}\right|_{\boldsymbol{V_2=0}},\ y_{22}(j\omega) = \left.\frac{\boldsymbol{I_2}}{\boldsymbol{V_2}}\right|_{\boldsymbol{V_1=0}}$$

for sinusoidal steady-state analysis of dynamic circuits with bold-faced italic quantities representing phasors.

Further, they get modified as

$$y_{11}(s) = \left.\frac{I_1(s)}{V_1(s)}\right|_{V_2(s)=0},\ y_{12}(s) = \left.\frac{I_1(s)}{V_2(s)}\right|_{V_1(s)=0},\ y_{21}(s) = \left.\frac{I_2(s)}{V_1(s)}\right|_{V_2(s)=0},\ y_{22}(s) = \left.\frac{I_2(s)}{V_2(s)}\right|_{V_1(s)=0}$$

for analysis of dynamic circuits in s-domain.

s-domain description is the most general description for a linear time-invariant circuit – memoryless or dynamic. Hence, we use s-domain description for two-port network parameters from this point onwards. Further, we express the two-port describing equation in matrix form as $\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix}$ where $\begin{bmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{bmatrix}$ is the *Short-Circuit Admittance Function Matrix* for the two-port network.

Determination of the four *y-parameters* requires us to solve two circuit problems. These two circuits are shown in Fig. 16.1-5.

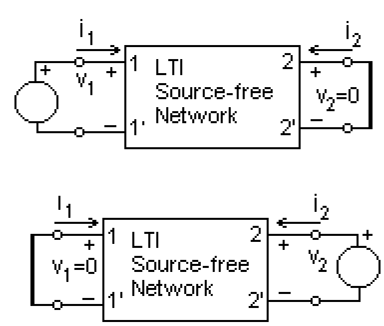

Fig. 16.1-5 Circuits to be solved for obtaining y-parameters

The first circuit has to be solved for i_1 and i_2 by employing standard circuit analysis procedures. Once these two variables have been obtained, we can work out y_{11} and y_{21} from them. Similarly solution for i_1 and i_2 by using standard circuit analysis procedure on the second circuit yields y_{22} and y_{12}. Obviously, the circuit within the box must be known completely for carrying out the required circuit analysis.

Open-Circuit Impedance Parameters for a Two-Port Network

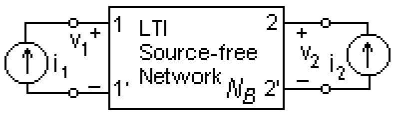

Fig. 16.1-6 Circuit in Fig. 16.1-3 reduced to a linear two-port driven by two current sources

The cascaded network in Fig. 16.1-2 was reduced to Fig. 16.1-4 for the purpose of relating the port-voltage variables and port-current variables of the two-port network in the previous sub-section. We were a bit too partial in deciding to replace the left-side and right-side networks by independent voltage sources. We could have used independent current sources too for this purpose. We apply Substitution theorem on the circuit in Fig. 16.1-2 and replace the networks N_A and N_C by suitable independent current sources now to arrive at the circuit shown in Fig. 16.1-6.

We can now express the port-voltage variables v_1 and v_2 as linear combinations of excitation functions i_1 and i_2 since the network is a linear one. This leads to the next description of a two-port network. This description will involve impedance functions in *s*-domain since a current excitation function is transformed into a voltage response function only through an impedance function. Therefore, the required describing equations in *s*-domain will be,

Defining equation for *z-parameters*

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} \quad (16.1\text{-}3)$$

where $\begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix}$ is the *Open-Circuit Impedance Parameter Matrix* for the two-port network. That they are *open-circuit parameters* is evident from the defining equations for the parameters as shown below.

$$z_{11}(s) = \left.\frac{V_1(s)}{I_1(s)}\right|_{I_2(s)=0}, z_{12}(s) = \left.\frac{V_1(s)}{I_2(s)}\right|_{I_1(s)=0}, z_{21}(s) = \left.\frac{V_2(s)}{I_1(s)}\right|_{I_2(s)=0}, z_{22}(s) = \left.\frac{V_2(s)}{I_2(s)}\right|_{I_1(s)=0}$$

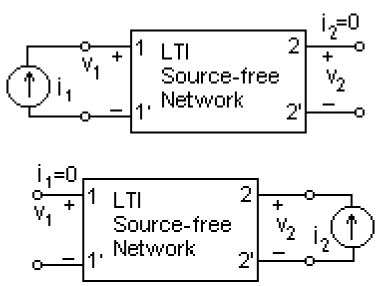

Fig. 16.1-7 Circuits to be solved for determining z-parameters

These are called *z-parameters* in shortened form. Determination of the four *z-parameters* requires us to solve two circuit problems. These two circuits are shown in Fig. 16.1-7.

The first circuit has to be solved for v_1 and v_2 by employing standard circuit analysis procedures. Once these two variables have been obtained we can work out z_{11} and z_{21} from them. Similarly solution for v_1 and v_2 by using standard circuit analysis procedure on the second circuit yields z_{22} and z_{12}.

Example : 16.1-1

Find the *y-parameters* and *z-parameters* for the resistive linear time-invariant two-port network in Fig. 16.1-8 with R_1 = 2Ω , R_2 = 4Ω, R_3 = 4Ω and R_4 = 1Ω.

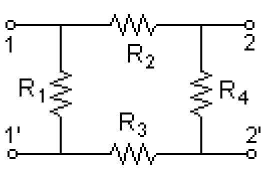

Fig. 16.1-8 Circuit for Example : 16.1-1

Solution

We determine *y-parameters* first. The circuit for determining y_{11} and y_{21} is shown in circuit (a) in Fig. 16.1-9.

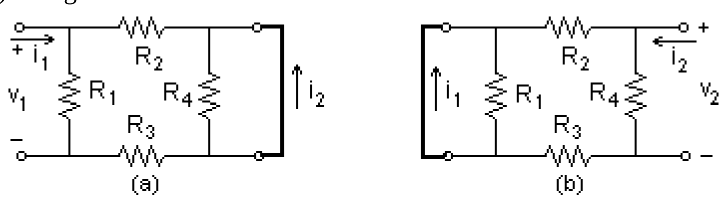

Fig. 16.1-9 Circuits for determining y-parameters in Example : 16.1-1

$$i_1 = \frac{v_1}{R_1} + \frac{v_1}{R_2 + R_3} = \frac{v_1}{2\Omega} + \frac{v_1}{8\Omega} = v_1 \times 0.625S\,;\; i_2 = -\frac{v_1}{R_2 + R_3} = -\frac{v_1}{8\Omega} = -v_1 \times 0.125S$$

$$\therefore y_{11} = \left.\frac{i_1}{v_1}\right|_{v_2=0} = 0.625S \text{ and } y_{21} = \left.\frac{i_2}{v_1}\right|_{v_2=0} = -0.125S$$

The circuit for determining y_{12} and y_{22} is shown in circuit (b) in Fig. 16.1-9.

$$i_2 = \frac{v_2}{R_4} + \frac{v_2}{R_2 + R_3} = \frac{v_2}{1\Omega} + \frac{v_2}{8\Omega} = v_2 \times 1.125S\,;\; i_1 = -\frac{v_2}{R_2 + R_3} = -\frac{v_2}{8\Omega} = -v_2 \times 0.125S$$

z_{11} is the input impedance at port-1 with port-2 kept open.

z_{12} is the reverse transfer impedance from port-2 to port-1 with port-1 kept open.

z_{22} is the input impedance at port-2 with port-1 kept open.

z_{21} is the forward transfer impedance from port-1 to port-2 with port-2 kept open.

$\therefore y_{12} = \left.\dfrac{i_1}{v_2}\right|_{v_1=0} = -0.125\,S$ and $y_{22} = \left.\dfrac{i_2}{v_2}\right|_{v_1=0} = 1.125\,S$

We determine *z-parameters* next. The circuit for determining z_{11} and z_{21} is shown in circuit (a) in Fig. 16.1-10.

$$v_1 = [R_1 // (R_2 + R_3 + R_4)] i_1 = i_1 \times 2\Omega // 9\Omega = i_1 \times 1.6364\Omega$$

$$v_2 = R_4 \times \frac{R_1}{R_1 + R_2 + R_3 + R_4} \times i_1 = i_1 \times 1\Omega \times \frac{2\Omega}{11\Omega} = i_1 \times 0.1818\Omega$$

$$\therefore z_{11} = \left.\frac{v_1}{i_1}\right|_{i_2=0} = 1.6364\Omega \text{ and } z_{21} = \left.\frac{v_2}{i_1}\right|_{i_2=0} = 0.1818\Omega$$

The circuit for determining z_{12} and z_{22} is shown in circuit (b) in Fig. 16.1-10.

$$v_2 = [R_4 // (R_2 + R_3 + R_1)] i_2 = i_2 \times 1\Omega // 10\Omega = i_2 \times 0.9091\Omega$$

$$v_1 = R_1 \times \frac{R_4}{R_1 + R_2 + R_3 + R_4} \times i_2 = i_2 \times 2\Omega \times \frac{1\Omega}{11\Omega} = i_2 \times 0.1818\Omega$$

$$\therefore z_{12} = \left.\frac{v_1}{i_2}\right|_{i_1=0} = 0.1818\Omega \text{ and } z_{22} = \left.\frac{v_2}{i_2}\right|_{i_1=0} = 0.9091\Omega$$

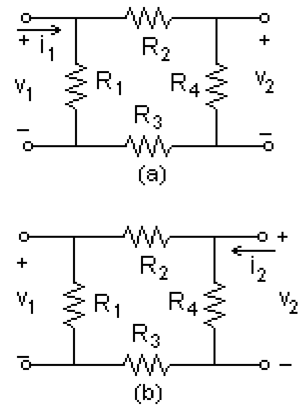

Fig. 16.1-10 Circuits for determining z-parameters in Example : 16.1-1

Therefore, the Short-Circuit Admittance Parameter Matrix is $\begin{bmatrix} 0.625 & -0.125 \\ -0.125 & 1.125 \end{bmatrix}$ Siemens and the Open-Circuit Impedance Parameter Matrix is $\begin{bmatrix} 1.6364 & 0.1818 \\ 0.1818 & 0.9091 \end{bmatrix}$ Ohms. The corresponding describing equations for the linear time-invariant two-port network is $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0.625 & -0.125 \\ -0.125 & 1.125 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1.6364 & 0.1818 \\ 0.1818 & 0.9091 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$.

We note that (i) y-matrix and z-matrix are inverses of each other (ii) $y_{12} = y_{21}$ and $z_{12} = z_{21}$ in this example.

Example : 16.1-2

The two-port network in Example : 16.1-1 is terminated in a resistor of value R Ω at its second port. Obtain an expression for its input resistance using *z-parameters*.

Solution

The describing equation in terms of *z-parameters* is

$$v_1 = z_{11} i_1 + z_{12} i_2$$
$$v_2 = z_{21} i_1 + z_{22} i_2$$

Connecting a resistor R across the output port imposes a constraint among v_2 and i_2. This constraint equation is given by $v_2 = -R i_2$. Substituting this in the second *z-parameter* equation, we get,

$$-R i_2 = z_{21} i_1 + z_{22} i_2 \Rightarrow i_2 = -\frac{z_{21}}{R + z_{22}} i_1$$

Substituting this expression for i_2 in the first *z-parameter* equation, we get,

$$v_1 = z_{11} i_1 + z_{12} i_2 = z_{11} i_1 - \frac{z_{12} z_{21}}{R + z_{22}} i_1 = i_1 [z_{11} - \frac{z_{12} z_{21}}{R + z_{22}}]$$

Therefore, input resistance is given by

$$R_{in} = \frac{v_1}{i_1} = [z_{11} - \frac{z_{12} z_{21}}{R + z_{22}}] = 1.6364 - \frac{0.1818^2}{R + 0.9091} = \frac{1.4546 + 1.6364R}{0.9091 + R}\,\Omega$$

Example : 16.1-3

(i) Find the *y-parameters* in *s*-domain for the circuit shown in Fig. 16.1-11. (ii) Obtain the transfer function between port-voltages when the second port is terminated in a resistance of R Ω.

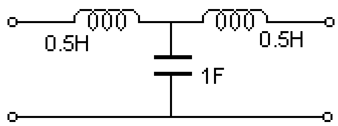

Fig. 16.1-11 Circuit for Example : 16.1-3

Solution

(i) The circuit needed for determining $y_{11}(s)$ and $y_{21}(s)$ is shown in circuit (a) of Fig. 16.1-12.

$$V_1(s) = \left[0.5s + \frac{0.5s \times 1/s}{0.5s + 1/s}\right] \times I_1(s) = \frac{s(0.25s^2+1)}{(0.5s^2+1)} \times I_1(s)$$

$$\therefore I_1(s) = \frac{(0.5s^2+1)}{s(0.25s^2+1)} \times V_1(s)$$

$$\text{and } I_2(s) = -\frac{1/s}{0.5s+1/s} \times \frac{(0.5s^2+1)}{s(0.25s^2+1)} \times V_1(s) = -\frac{1}{s(0.25s^2+1)} \times V_1(s)$$

$$\therefore y_{11}(s) = \frac{(0.5s^2+1)}{s(0.25s^2+1)} S \text{ and } y_{21}(s) = -\frac{1}{s(0.25s^2+1)} S$$

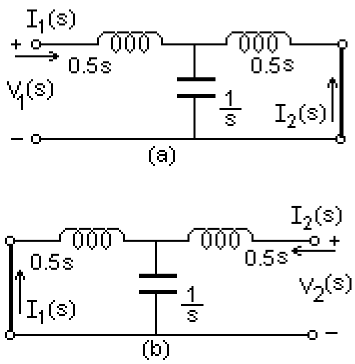

Fig. 16.1-12 Circuits for determining y-parameters in Example : 16.1-3

We observe that there is essentially no difference between the circuit (a) in Fig. 16.1-12 and circuit (b) in the same figure except that the excitation and response ports are interchanged. Thus, we expect that $y_{22}(s)$ will be same as $y_{11}(s)$ and $y_{12}(s)$ will be same as $y_{21}(s)$.

$$\therefore y_{22}(s) = \frac{(0.5s^2+1)}{s(0.25s^2+1)} S \text{ and } y_{12}(s) = -\frac{1}{s(0.25s^2+1)} S$$

(ii) Connecting a resistor R across the output port imposes a constraint between $V_2(s)$ and $I_2(s)$. This constraint equation is $V_2(s) = -RI_2(s)$. Substituting this constraint equation in the second *y-parameter* equation, we get,

$$I_2(s) = y_{21}(s)V_1(s) + y_{22}(s)V_2(s)$$
$$= y_{21}(s)V_1(s) - Ry_{22}(s)I_2(s)$$

$$\therefore I_2(s) = \frac{y_{21}(s)V_1(s)}{1+Ry_{22}(s)}$$

$$\therefore V_2(s) = -RI_2(s) = \frac{-Ry_{21}(s)V_1(s)}{1+Ry_{22}(s)}$$

$$\therefore H(s) = \frac{V_2(s)}{V_1(s)} = \frac{-Ry_{21}(s)}{1+Ry_{22}(s)}$$

Substituting for $y_{22}(s)$ and $y_{21}(s)$, we get the transfer function as

$$H(s) = \frac{4R}{s^3 + s^2(2R) + 4s + 4R} \text{ V/V}$$

Example : 16.1-4

Find the *z-parameters* of the two-port network shown in Fig. 16.1-13.

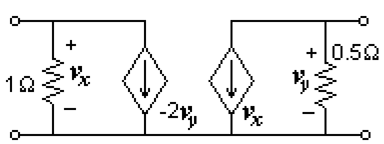

Fig. 16.1-13 Circuit for Example : 16.1-4

Solution

The circuit to be analysed to determine z_{11} and z_{21} is shown as (a) in Fig. 16.1-14.
We solve this circuit to obtain v_1 and v_2 in terms of i_1.
Applying KCL at first terminal at left-side port in circuit (a), we get,

$$i_1 = \frac{v_x}{1} - 2v_y \text{ .}$$

Further, $v_y = -0.5v_x$ by Ohm's law for the 0.5Ω resistor across the second port. Combining these two equations, we get,

$$i_1 = v_x - 2(-0.5v_x) = 2v_x$$

But $v_x = v_1$ and hence $z_{11} = \left.\frac{v_1}{i_1}\right|_{i_2=0} = 0.5\Omega$

Now we have $v_y = -0.5v_x$ and $i_1 = 2v_x$.

Therefore, $v_2 = v_y = -0.25i_1$ and hence $z_{21} = \left.\frac{v_2}{i_1}\right|_{i_2=0} = -0.25\Omega$.

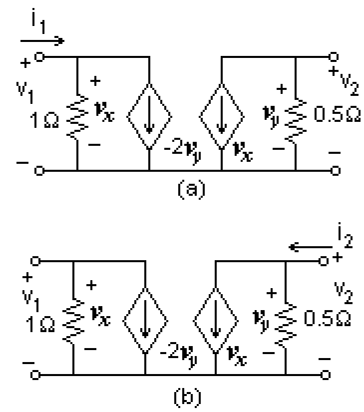

Fig. 16.1-14 Circuits for determination of z-parameters in Example : 16.1-4

The circuit to be analysed to determine z_{11} and z_{21} is shown as (b) in Fig. 16.1-14. We solve this circuit to obtain v_1 and v_2 in terms of i_2.

Applying KCL at first terminal at right-side port in circuit (b), we get,

$i_2 = v_x + 2v_y$.

Further, $v_x = 2v_y$ by Ohm's law for the 1Ω resistor across the first port. Combining these two equations, we get,

$i_2 = 2v_y + 2v_y = 4v_y$

But $v_2 = v_y$ and hence $z_{22} = \left.\frac{v_2}{i_2}\right|_{i_1=0} = 0.25\Omega$

Now we have $v_x = 2v_y$ and $i_2 = 4v_y$.

Therefore, $v_1 = v_x = 0.5i_2$ and hence $z_{12} = \left.\frac{v_1}{i_2}\right|_{i_1=0} = 0.5\Omega$.

Therefore, the Open-Circuit Impedance Matrix of this two-port network is $\begin{bmatrix} 0.5 & 0.5 \\ -0.25 & 0.25 \end{bmatrix}\Omega$.

Hybrid Parameters and Inverse-Hybrid Parameters for a Two-Port Network

Refer to Fig. 16.1-2. We apply Substitution theorem to replace the network N_A by a current source of value i_1 first. Substitution theorem is applied again to the resulting network to replace the network N_C by a voltage source of value v_2 to arrive at the reduced network shown in Fig. 16.1-15.

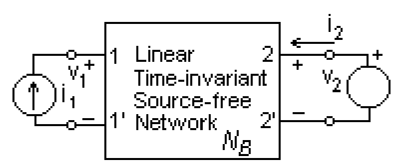

Fig. 16.1-15 Circuit in Fig. 16.1-3 reduced to a two-port circuit driven by a current source and voltage source

We can now express the input port-voltage variable v_1 and output port-current variable i_2 as linear combinations of excitation functions i_1 and v_2 since the network is a linear one. This leads to the next description of a two-port network. This description will involve impedance function, admittance function and transfer functions in *s*-domain since it is a mixed excitation set and a mixed response set. Therefore, the required describing equations in *s*-domain will be,

Defining equation for *h-parameters*

$$\begin{bmatrix} V_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} h_{11}(s) & h_{12}(s) \\ h_{21}(s) & h_{22}(s) \end{bmatrix} \begin{bmatrix} I_1(s) \\ V_2(s) \end{bmatrix}$$

where $\begin{bmatrix} h_{11}(s) & h_{12}(s) \\ h_{21}(s) & h_{22}(s) \end{bmatrix}$ is the *Hybrid Parameter Matrix* for the two-port network. The defining equations for the parameters are as shown below.

$$h_{11}(s) = \left.\frac{V_1(s)}{I_1(s)}\right|_{V_2(s)=0}, h_{12}(s) = \left.\frac{V_1(s)}{V_2(s)}\right|_{I_1(s)=0}, h_{21}(s) = \left.\frac{I_2(s)}{I_1(s)}\right|_{V_2(s)=0}, h_{22}(s) = \left.\frac{I_2(s)}{V_2(s)}\right|_{I_1(s)=0}$$

These are called *h-parameters* in shortened form. h_{11}(s) is the input impedance function with output port shorted. h_{12}(s) is the *reverse voltage transfer ratio with input open* and h_{21}(s) is the *forward current transfer ratio with output shorted*. h_{22}(s) is the admittance seen from second port with first port kept open.

Determination of the four *h-parameters* requires us to solve two circuit problems. These two circuits are shown in Fig. 16.1-16.

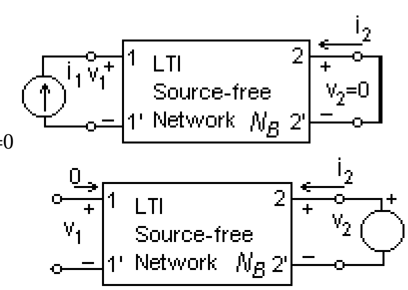

Fig. 16.1-16 Circuits that are to be solved for determination of h-parameters

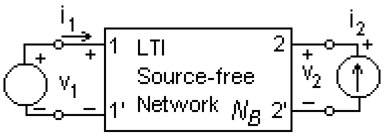

Fig. 16.1-17 Circuit in Fig. 16.1-3 reduced to a two-port network driven by a voltage source and current source

Refer to Fig. 16.1-2 again. We apply Substitution theorem to replace the network N_A by a voltage source of value v_1 first. Substitution theorem is applied again to the resulting network to replace the network N_C by a current source of value i_2 to arrive at the reduced network shown in Fig. 16.1-17.

We can now express the input port-current variable i_1 and output port-voltage variable v_2 as linear combinations of excitation functions v_1 and i_2 since the network is a linear one. This leads to the next description of a two-port network. This description will involve impedance function, admittance function and transfer functions in *s*-domain since it is a mixed excitation set and a mixed response set. Therefore, the required describing equations in *s*-domain will be,

$$\begin{bmatrix} I_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \begin{bmatrix} V_1(s) \\ I_2(s) \end{bmatrix}$$

where $\begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}$ is the *Inverse-Hybrid Parameter Matrix* for the two-port network. The defining equations for the parameters are as shown below.

$$g_{11}(s) = \left.\frac{I_1(s)}{V_1(s)}\right|_{I_2(s)=0}, g_{12}(s) = \left.\frac{I_1(s)}{I_2(s)}\right|_{V_1(s)=0}, g_{21}(s) = \left.\frac{V_2(s)}{V_1(s)}\right|_{I_2(s)=0}, g_{22}(s) = \left.\frac{V_2(s)}{I_2(s)}\right|_{V_1(s)=0}$$

These are called *g-parameters* in shortened form. Determination of the four *g-parameters* requires us to solve two circuit problems. These two circuits are shown in Fig. 16.1-18.

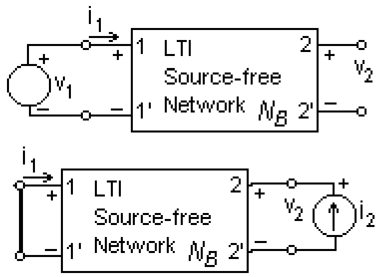

Fig. 16.1-18 Circuits that are to be solved for determination of g-parameters

Example : 16.1-5

Find the *h-parameters* and *g-parameters* of the two-port network shown in Fig. 16.1-19. R_s = 1kΩ, R_i = 98.01kΩ, R_o = 1kΩ, R_1 = 1kΩ, R_2 = 99kΩ and K = 100.

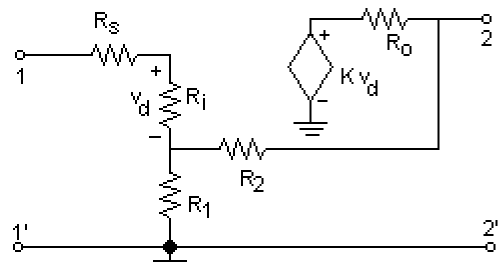

Fig. 16.1-19 Circuit for Example : 16.1-5

h_{11}(s) is the input impedance function with output port shorted.

h_{12}(s) is the *reverse voltage transfer ratio* with input port open.

h_{21}(s) is the *forward current transfer ratio* with output port shorted.

h_{22}(s) is the admittance seen from output port with input port kept open.

Solution

We determine *h-parameters* first. The defining equations of *h-parameters* for a memoryless two-port network are

$$h_{11} = \left.\frac{v_1}{i_1}\right|_{v_2=0} ; h_{21} = \left.\frac{i_2}{i_1}\right|_{v_2=0} ; h_{12} = \left.\frac{v_1}{v_2}\right|_{i_1=0} \text{ and } h_{22} = \left.\frac{i_2}{v_2}\right|_{i_1=0}.$$

The circuit to be analysed for determining h_{11} and h_{21} is shown as circuit (a) in Fig. 16.1-20. And, the circuit to be analysed for determining h_{12} and h_{22} is shown as circuit (b) in the same figure.

g_{11}(s) is the input admittance function with output port open.

g_{12}(s) is the *reverse current transfer ratio* with input port shorted .

g_{21}(s) is the *forward voltage transfer ratio* with output port open.

g_{22}(s) is the impedance seen from output port with input port kept shorted.

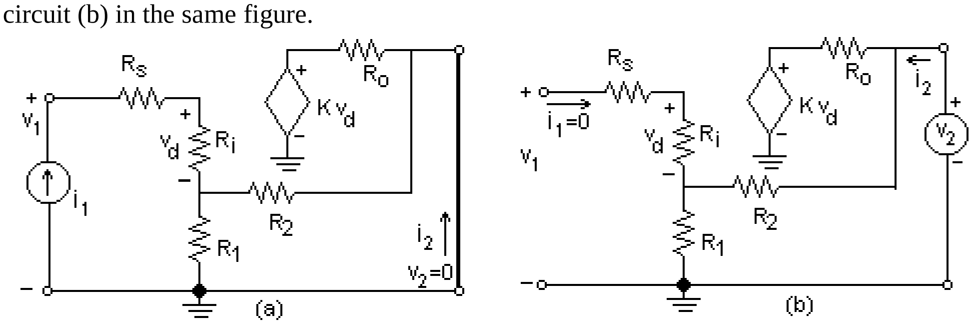

Fig. 16.1-20 Circuits to be solved for determining h-parameters in Example : 16.1-5

We take up circuit (a) first. The short across second port connects the resistor R_2 directly across R_1.

$$\therefore v_1 = i_1 \times [R_s + R_i + R_1 // R_2] = i_1 \times [1 + 98.01 + 1 // 99]\text{kÙ}$$
$$\therefore h_{11} = [1 + 98.01 + 1 // 99]\text{kÙ} = 100\text{kÙ}$$

Further, the voltage v_d across R_i is $R_i i_1$ and hence the dependent source at the second port produces $K\,R_i i_1$ volts across its output. This source delivers $K\,R_i i_1/R_o$ amps of current into the short across the second port. The second component of current in the short is the one coming from the R_2 line. Its value is $i_1 \times R_1/(R_1+R_2)$. Therefore,

$$i_2 = -\left(\frac{i_1 R_1}{R_1 + R_2} + \frac{KR_i i_1}{R_o}\right) \Rightarrow h_{21} = -\left(\frac{R_1}{R_1 + R_2} + \frac{KR_i}{R_o}\right) = -9801.01$$

Now consider the circuit (b) in Fig. 16.1-20. Since the first port is left open, the voltage v_d is zero. Hence the dependent voltage source at the second port is a short. Therefore, the resistor R_o and series combination of R_1 and R_2 come directly across the voltage source v_2.

$$\therefore h_{22} = \left.\frac{i_2}{v_2}\right|_{i_1=0} = \frac{1}{R_o} + \frac{1}{R_1 + R_2} = 0.00101S$$

Further, since i_1 is zero, there is no voltage drop across R_s and R_i. Therefore v_1 is the same as the voltage across the resistor R_1.

$$\therefore v_1 = \frac{R_1}{R_1 + R_2} v_2 \Rightarrow h_{12} = \frac{R_1}{R_1 + R_2} = 0.01$$

Therefore, the *h-parameter* matrix for this linear time-invariant two-port network is given by $\begin{bmatrix} 0.1\text{MÙ} & 0.01 \\ -9801.01 & 1.01\text{mS} \end{bmatrix}$.

We determine *g-parameters* next. The defining equations of *g-parameters* for a memoryless two-port network are $g_{11} = \left.\frac{i_1}{v_1}\right|_{i_2=0}$; $g_{21} = \left.\frac{v_2}{v_1}\right|_{i_2=0}$; $g_{12} = \left.\frac{i_1}{i_2}\right|_{v_1=0}$ and $g_{22} = \left.\frac{v_2}{i_2}\right|_{v_1=0}$. The circuit to be analysed for determining g_{11} and g_{21} is shown as circuit (a) in Fig. 16.1-21. And, the circuit to be analysed for determining g_{12} and g_{22} is shown as circuit (b) in the same figure. Note that the dependent voltage source in series with R_o at the second port has been converted into a dependent current source in parallel with R_o in both circuits.

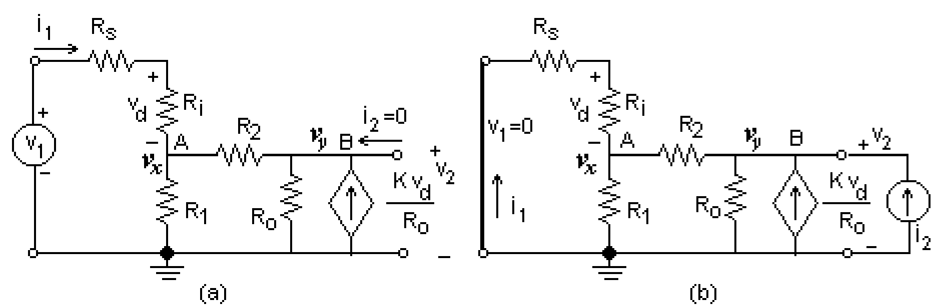

Fig. 16.1-21 Circuits to be solved for determination of g-parameters in Example : 16.1-5

We take up the circuit (a) in Fig. 16.1-21 first. Two nodes *A* and *B*and corresponding node voltage variables v_x and v_y are identified in this circuit. We write the node equations at these two nodes.

$$\text{Node A} : v_x\left[\frac{1}{R_s + R_i} + \frac{1}{R_2} + \frac{1}{R_1}\right] - v_y \frac{1}{R_2} = v_1 \frac{1}{R_s + R_i}$$
$$\text{Node B} : v_y\left[\frac{1}{R_2} + \frac{1}{R_o}\right] - v_x \frac{1}{R_2} = K \frac{R_i}{R_i + R_s} \frac{(v_1 - v_x)}{R_o}$$

Note that we have substituted for v_d in terms of v_1 and v_x using voltage division principle relevant to series connected resistors. Substituting the component values and simplifying the expressions, we get,

$$\begin{bmatrix} 1.0201 & -0.01 \\ 98.99 & 1.01 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0.0101 \\ 99 \end{bmatrix} v_1 .$$

Solving for the node voltage variables, we get,

$$v_x = 0.495 v_1 \text{ and } v_y = 49.5 v_1$$

$$\therefore v_2 = v_y = 49.5 v_1 \Rightarrow g_{21} = 49.5$$

$$\text{and } i_1 = \frac{v_1 - v_x}{R_s + R_i} = \frac{v_1 - 0.495 v_1}{R_s + R_i} = \frac{0.505 v_1}{R_s + R_i} \Rightarrow g_{11} = \frac{0.505}{R_s + R_i} = 5.1 \times 10^{-6}\ S$$

Now we consider circuit (b) in Fig. 16.1-21. The node voltage variables assigned are v_x and v_y at node A and node B respectively. We write the node equations as

$$\text{Node A} : v_x \left[\frac{1}{R_s + R_i} + \frac{1}{R_2} + \frac{1}{R_1} \right] - v_y \frac{1}{R_2} = 0$$

$$\text{Node B} : v_y \left[\frac{1}{R_2} + \frac{1}{R_o} \right] - v_x \frac{1}{R_2} = i_2 - K \frac{R_i}{R_i + R_s} \frac{v_x}{R_o}$$

Note that we have substituted for v_d in terms of v_x using voltage division principle relevant to series connected resistors.

Substituting the component values and simplifying the expressions, we get,

$$\begin{bmatrix} 1.0201 & -0.01 \\ 98.99 & 1.01 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 \\ 1000 i_2 \end{bmatrix}$$

Solving for the node voltage variables, we get,

$$v_x = 4.95 i_2 \text{ and } v_y = 504.95 i_2 \Rightarrow v_2 = 504.95 i_2 \Rightarrow g_{22} = 504.95\Omega$$

$$\text{Further, } i_1 = -\frac{v_x}{R_s + R_i} = -\frac{4.95 i_2}{R_s + R_i} = -5 \times 10^{-5} i_2 \Rightarrow g_{12} = -5 \times 10^{-5}$$

Therefore, the *g-parameter* matrix for this linear time-invariant two-port network is given by $\begin{bmatrix} 5.1\mu\text{S} & -5 \times 10^{-5} \\ 49.5 & 504.95\Omega \end{bmatrix}$.

16.2 Equivalent Circuits for a Two-Port Network

The describing equations for a memoryless linear time-invariant two-port network using *y-parameters* is given by

$$\begin{aligned} i_1 &= y_{11} v_1 + y_{12} v_2 \\ i_2 &= y_{21} v_1 + y_{22} v_2 \end{aligned} .$$

Interpreting two-port parameter defining equations to arrive at an equivalent circuit for the network

The left-sides of the equations are currents. Hence the terms in the right-sides must also be currents. Consider the first equation. We can see that the term $y_{11}v_1$ represents the current drawn by an admittance of y_{11} from the port-voltage v_1. Hence, this term can be accounted for by connecting an admittance of y_{11} across the first port. However, the second term represents a current drawn from first port with the value of this current decided by the *second port-voltage*. Therefore, it can not be taken into account by connecting any admittance across the first port. But then, it can be accounted for by connecting a dependent source – a voltage-controlled current source across the first port with second port-voltage v_2 as the controlling variable. Thus, an admittance of y_{11} in parallel with a dependent current source at the input port will satisfy the first equation. The second equation too can be interpreted in a similar manner as the parallel combination of an admittance of y_{22} and a dependent current source $y_{21}i_1$ across the second port. This interpretation leads to an equivalent circuit shown in (a) of Fig. 16.2-1 for the linear time-invariant two-port network. The voltage and current variables are instantaneous variables and the parameters are real numbers for a memoryless two-port. The voltage and current variables are Laplace transforms and parameters are ratio of rational polynomials in s for a general linear time-invariant dynamic circuit.

Similar interpretations for the remaining three descriptions of a linear time-invariant two-port network in terms of *z, h* and *g parameters* lead to three other equivalent circuits shown in (b), (c) and (d) of Fig. 16.2-1 respectively.

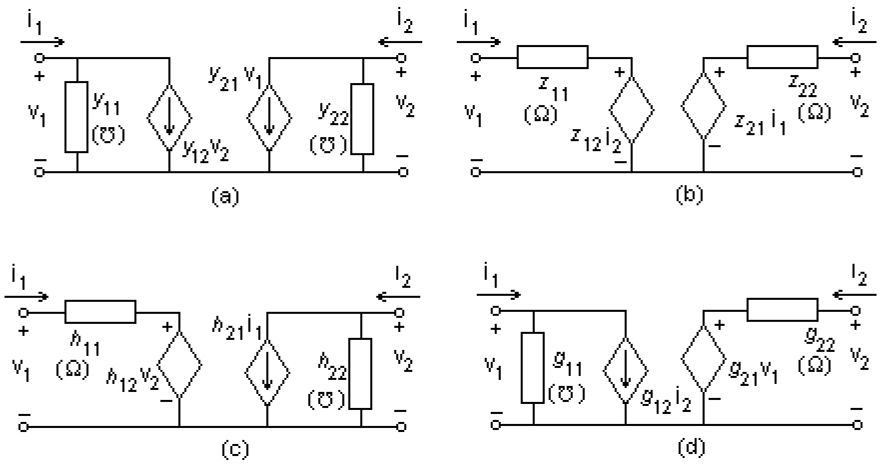

Fig. 16.2-1 Equivalent Circuits for a Linear Time-Invariant Two-Port Network

But this does not imply that any given two-port will have all the four parameter sets and corresponding equivalent circuits. These equivalent circuits exist only if the corresponding parameters exist. It is possible that a given linear time-invariant two-port network has only some parameter sets. For instance, the circuit (a) in Fig. 16.2-2 has no *z and h parameters* (*i.e.,* all *z-parameters* are infinite) but has *y and g parameters*. The circuit (b) in Fig. 16.2-2 has no *y, h and g* parameters; but it has *z-parameters*.

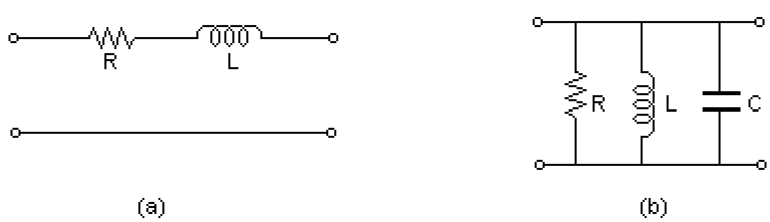

Fig. 16.2-2 (a) A Two-Port with no z-parameters (b) A Two-Port with no y-parameters

16.3 Transmission Parameters (ABCD Parameters) of a Two-Port Network

Two-port network theory finds application in solving transmission line problems in Electrical and Electronics Engineering. Power System Engineers often have to solve for the magnitude of sending end voltage to be maintained in the line in order to ensure a pre-specified voltage magnitude at the receiving end when the receiving end is delivering a pre-specified power at a certain power factor. In addition, they would like to solve for the sending end line current, sending end active and reactive power etc., under this condition. The problem essentially involves determination of v_1 and i_1 of a two-port network given v_2 and i_2 of the network. A similar problem arises in filtering context. The filter engineer wants to study the attenuation and phase shift suffered by a signal when it goes through a linear time-invariant two-port network.

The four parameter-sets we discussed in the previous sub-sections are the *basic parameter sets* of a two-port. None of the *basic parameter sets* are suited for the analysis problem described above. A new parameter set, which is a *derived parameter set* is used for this kind of *transmission analysis problems*. This new parameter set is called '*Transmission Parameter Set*'or *ABCD Parmeters* for the two-port. One method to arrive at this set is shown below.

We want to express v_1 and i_1 in terms of v_2 and i_2. We start with *y-parameter* description of the two-port.

$$i_1 = y_{11}v_1 + y_{12}v_2$$
$$i_2 = y_{21}v_1 + y_{22}v_2$$

Algebraic manipulation of the second equation leads to $v_1 = -\frac{y_{22}}{y_{21}}v_2 + \frac{1}{y_{21}}i_2$.

Substituting this in first equation, we get, $i_1 = \left(y_{12} - \frac{y_{11}y_{22}}{y_{21}}\right)v_2 + \frac{y_{11}}{y_{21}}i_2$. Thus the required expressions that express the first port-variables in terms of second port-variables are,

$$v_1 = -\frac{y_{22}}{y_{21}}v_2 + \frac{1}{y_{21}}i_2$$
$$i_1 = \left(y_{12} - \frac{y_{11}y_{22}}{y_{21}}\right)v_2 + \frac{y_{11}}{y_{21}}i_2$$

A passively terminated two-port will usually *deliver* current into the passive load connected across second port. Therefore, it is convenient to consider (-i_2) as the port-current variable of interest in analysing transmission problems. Hence, the above set of equations have been traditionally written as

$$\begin{aligned} v_1 &= Av_2 + B(-i_2) \\ i_1 &= Cv_2 + D(-i_2) \end{aligned}$$ or as $\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$ in matrix form. We see that $A = -\frac{y_{22}}{y_{21}}, B = -\frac{1}{y_{21}}, C = \left(\frac{y_{12}y_{21} - y_{11}y_{22}}{y_{21}}\right)$ and $D = -\frac{y_{11}}{y_{21}}$. We could have started with *z-parameter* description and arrived at equivalent expressions for transmission parameters in terms of *z-parameters*. Similarly, *ABCD parameters* can be expressed in terms of *h* or *g parameters* too.

Note that the equation $\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$ does not imply that the variables v_2 and i_2 are the independent variables, *i.e.*, sources and the variables v_1 and i_1 are response variables. Obviously, both the port-voltage and port-current of a port can not be constrained by independent voltage source and independent current source respectively at the same time. Thus, *ABCD parameter* description of a two-port is to be understood as a doctored form of one of the four *basic parameter descriptions* for the same two-port. In this sense the *ABCD parameter set* is a derived set and is not a basic set. *ABCD parameter* based description of a two-port network does not yield an equivalent circuit for it for the same reason.

The transmission parameters can be interpreted as follows.

$$A = \text{Reverse Voltage Transfer Ratio} = \left.\frac{v_1}{v_2}\right|_{i_2=0}$$

$$B = \text{Short-Circuit Transfer Impedance} = \left.\frac{v_1}{(-i_2)}\right|_{v_2=0} \quad (\Omega)$$

$$C = \text{Open-Circuit Transfer Admittance} = \left.\frac{i_1}{v_2}\right|_{i_2=0} \quad (S)$$

$$D = \text{Reverse Current Transfer Ratio} = \left.\frac{i_1}{(-i_2)}\right|_{v_2=0}$$

All these parameters are ratios of rational polynomials in *s* for a general dynamic two-port network.

Example : 16.3-1

A 50 Hz , 250 kM transmission line can be modeled approximately by the Π-equivalent circuit shown in Fig. 16.3-1. The receiving end load is 40 MW at 220kV with 0.8 lagging power factor. (i) Find *ABCD* parameters of the line. (ii) Find the sending-end voltage magnitude and angle, sending-end current, active and reactive power at sending-end, sending-end power factor and line transmission efficiency using two-port transmission equations.

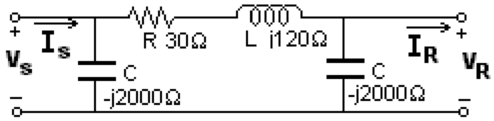

Fig. 16.3-1 Equivalent Circuit of Transmission Line in Example : 16.3-1

Solution

(i) We derive the *ABCD parameters* for the two-port network shown in Fig. 16.3-2 first and substitute the relevant numbers later. $\boldsymbol{Z}$ is the series phasor impedance and $\boldsymbol{Y}$ is the *total admittance* of the two shunt branches.

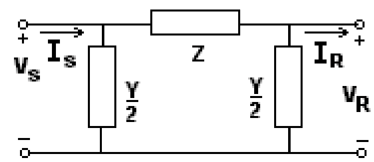

Fig. 16.3-2 A Π Two-Port Network

$$\begin{bmatrix} \boldsymbol{V_S} \\ \boldsymbol{I_S} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \boldsymbol{V_R} \\ \boldsymbol{I_R} \end{bmatrix}$$

The circuits to be solved for determination of *ABCD parameters* are shown in Fig. 16.3-3.

Solving the first circuit, we get, $\boldsymbol{V_R} = \boldsymbol{V_S} \frac{2/\boldsymbol{Y}}{\boldsymbol{Z} + 2/\boldsymbol{Y}} = \boldsymbol{V_S} \frac{1}{1 + \frac{1}{2}\boldsymbol{ZY}}$ and

$$\boldsymbol{I_S} = \boldsymbol{V_S}\left(\frac{\boldsymbol{Y}}{2} + \frac{1}{\boldsymbol{Z} + 2/\boldsymbol{Y}}\right) = \boldsymbol{V_S}\boldsymbol{Y}\left(\frac{4 + \boldsymbol{ZY}}{2(2 + \boldsymbol{ZY})}\right) = \boldsymbol{V_S}\boldsymbol{Y}\left(\frac{4 + \boldsymbol{ZY}}{4(1 + \frac{1}{2}\boldsymbol{ZY})}\right) = \boldsymbol{V_S}\boldsymbol{Y}\left(1 + \tfrac{1}{4}\boldsymbol{ZY}\right).$$

Comparing with the *ABCD* equations, we get, $A = 1 + \frac{1}{2}\boldsymbol{ZY}$ and $C = \boldsymbol{Y}\left(1 + \frac{1}{4}\boldsymbol{ZY}\right)$

Solving the second circuit in Fig. 16.3-3, we get,

$$\boldsymbol{I_R} = \boldsymbol{V_S}\frac{1}{\boldsymbol{Z}} \Rightarrow \boldsymbol{V_S} = \boldsymbol{Z}\boldsymbol{I_R}\ ,\ \boldsymbol{I_R} = \boldsymbol{I_S}\frac{2/\boldsymbol{Y}}{\boldsymbol{Z} + 2/\boldsymbol{Y}} = \boldsymbol{I_S}\frac{1}{1 + \frac{1}{2}\boldsymbol{ZY}} \Rightarrow \boldsymbol{I_S} = \boldsymbol{I_R}\left(1 + \tfrac{1}{2}\boldsymbol{ZY}\right)$$

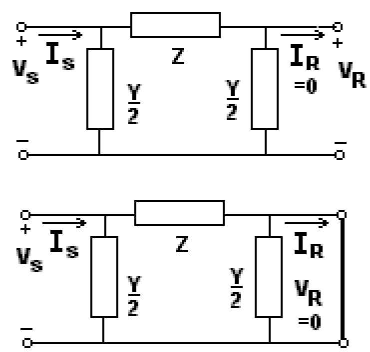

Fig. 16.3-3 Phasor Equivalent Circuits for Determination of ABCD Parameters in Example : 16.3-1

Comparing with the *ABCD* equations, we get, $B = \boldsymbol{Z}$ and $D = 1 + \frac{1}{2}\boldsymbol{ZY}$. Therefore the *ABCD* matrix for the Π-network in Fig. 16.3-2 is

$\begin{bmatrix} 1 + \frac{1}{2}\boldsymbol{ZY} & \boldsymbol{Z} \\ \boldsymbol{Y}\left(1 + \frac{1}{4}\boldsymbol{ZY}\right) & 1 + \frac{1}{2}\boldsymbol{ZY} \end{bmatrix}$. Substituting $\boldsymbol{Z} = 30 + j120\,\Omega$ and $\boldsymbol{Y} = j10^{-3}S$, we get,

$$\begin{bmatrix} 0.94\angle 0.91^0 & 123.69\angle 75.96^0\,\Omega \\ 0.00097\angle 90.44^0\,S & 0.94\angle 0.91^0 \end{bmatrix}.$$

(ii) The receiving-end active power is 40 MW and power factor is 0.8 lag. Therefore the receiving-end current magnitude is $40\times10^6/(0.8\times220\times10^3)$=227.27 amps. The phase angle of current with respect to voltage is $-\cos^{-1}(0.8) = -36.87^0$. Taking the receiving-end voltage as the reference phasor, we now have $\boldsymbol{V_R} = 220\times10^3\angle 0^0$ V and $\boldsymbol{I_R} = 227.27\angle -36.87^0$ A. Substituting these values in the *ABCD* equations, we have,

$$\begin{bmatrix} \boldsymbol{V_S} \\ \boldsymbol{I_S} \end{bmatrix} = \begin{bmatrix} 0.94\angle 0.91^0 & 123.69\angle 75.96^0\,\Omega \\ 0.00097\angle 90.44^0\,S & 0.94\angle 0.91^0 \end{bmatrix} \begin{bmatrix} 220\times10^3 \\ 227.27\angle -36.87^0 \end{bmatrix}$$

Therefore, $\boldsymbol{V_S} = 229.6\angle 5.26^0$ kV and $\boldsymbol{I_S} = 192.56\angle 27.18^0$ A. The sending-end power $\boldsymbol{S_S} = \boldsymbol{V_S}\boldsymbol{I_S}^* = (41.012\text{-}j16.504)$ MVA. Hence active power at sending-end = 41.012 MW , reactive power = -16.504 MVAr (*i.e.*, capacitive), power factor is $\cos(27.18^0 \sim 5.26^0)$=0.928 *lead* and transmission efficiency is = 100×40/41.012 = 97.32%.